Encoding/Decoding Strategies, Achievable Rates, and Resource Allocation for Cooperative Multiple Access Channels

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Information theoretic analysis of communication systems

- Provides a benchmark for system performance – fundamental limits
  - E.g., “how close are we to channel capacity achievable by any scheme?”

- Gives direction to future research
  - It tells you what could still be achieved.
  - It suggests new ways to push your limits
    * E.g., use of multiple antennas, user cooperation, etc.

- Though strictly theoretical, gives insight to practical algorithms and applications
  - E.g., how to do encoding, decoding, resource allocation, medium access, etc.
• The channel is memoryless if \( P(Y^n|X^n) = \prod_{i=1}^{n} P(Y|X) \).

• A communication channel has capacity \( C \), if
  – Any rate \( R < C \) can be transmitted reliably (i.e., with arbitrarily low probability of error).
  – Any rate \( R > C \) is guaranteed to have probability of error bounded away from zero.

• Achieved by using a random coding argument.

\[
C = \max_{p(x)} I(X; Y)
\]

• \( p(x) \) is the marginal distribution of the random variable \( X \).

• \( I(X; Y) \) is the mutual information between \( X \) and \( Y \), i.e., the reduction of uncertainty about \( X \) upon observing \( Y \).
• The channel is memoryless if $P(Y^n|X^n) = \prod_{i=1}^{n} P(Y|X)$.

• A communication channel has capacity $C$, if
  – Any rate $R < C$ can be transmitted reliably (i.e., with arbitrarily low probability of error).
  – Any rate $R > C$ is guaranteed to have probability of error bounded away from zero.

• Achieved by using a random coding argument.

$$C = \max_{p(x)} I(X;Y)$$

• $p(x)$ is the marginal distribution of the random variable $X$.

• $I(X;Y)$ is the mutual information between $X$ and $Y$, i.e., the reduction of uncertainty about $X$ upon observing $Y$. 
\[ Y[i] = X[i] + N[i], \quad i = 1, \ldots, N \quad (1) \]

- For Gaussian channels with signal power \( P \) and noise variance \( \sigma^2 \), the capacity is given by

\[
C = \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right)
\]

- To achieve capacity, the codeword \( X^n \) is taken from a codebook generated randomly,
  - Each symbol in the sequence \( X^n \) is i.i.d Gaussian, i.e., \( X \sim \mathcal{N}(0, P) \).

- The capacity is achieved as the codeword length \( n \to \infty \)

- Decoding is performed based on jointly typical sequences.
Multiple sources convey independent messages to the same receiver.
- E.g., uplink of a cellular system, all mobiles send data to the base station.

- The rates achievable by users is worse than their single user performance due to interference.
Region of achievable rates rather than a single rate value.

The capacity region is a pentagon

The rate of a user can be increased up to its single user limit, in expense of rate of other user.

Corners of the boundary can be achieved by successive decoding.
• **Fading**: random fluctuations in the channel.

• Known statistics and the realization of the fading ⇒ **opportunistic resource allocation**.

• **Power control**
  – Quality of service based (instantaneous requirements)
  – Capacity based (long term requirements)

• We are interested in long term capacities of systems with average power constraints.
Single User Channel (Goldsmith-Varaiya 1994)

- Channel capacity for single user

\[ C = \frac{1}{2} \log \left( 1 + \frac{p}{\sigma^2} \right) \]

- In the presence of fading, for a fixed channel state \( h \)

\[ y = \sqrt{p(h)h} + n \]

\[ C(h) = \frac{1}{2} \log \left( 1 + \frac{p(h)h}{\sigma^2} \right) \]

- Maximize the ergodic capacity, given an average power constraint

\[
\max_{\{p(h)\}} \quad E_h \left[ \log \left( 1 + \frac{p(h)h}{\sigma^2} \right) \right]
\]

s.t. \[ E_h [p(h)] \leq \bar{p}, \quad p(h) \geq 0 \]
• Optimal power allocation: waterfilling of power over time

\[ p(h) = \left( \frac{1}{\lambda} - \frac{\sigma^2}{h} \right)^+ \]

• More power to better channel states; no power to very poor channel states
The received signal
\[ y = \sum_{i=1}^{K} \sqrt{p_i(h)} h_i x_i + n \]

Region of achievable rates instead of a single capacity.

Maximize ergodic sum capacity, given average power constraints
\[
\max_{\{p_i(h)\}} \mathbb{E}_h \left[ \frac{1}{2} \log \left( 1 + \sigma^{-2} \sum_{i=1}^{K} h_i p_i(h) \right) \right]
\]
\[ \text{s.t.} \quad \mathbb{E}_h [p_i(h)] \leq \bar{p}_i, \quad p_i(h) \geq 0, \quad i = 1, \ldots, K \]

Optimal power allocation: single user waterfilling on disjoint sets of channel states
\[
p_k(h) = \begin{cases} \left( \frac{1}{\lambda_k} - \frac{\sigma^2}{h_k} \right)^+, & \text{if } h_k/\lambda_k > h_j/\lambda_j, \quad \forall j \neq k \\ 0, & \text{otherwise} \end{cases}
\]

Only the strongest user transmits at any given time. More than one user transmit w.p. 0.
Optimum Power Allocation: Scalar Multiuser Channel

Power Distribution of User 1

Power Distribution of User 2
Capacity Region of Fading Scalar MAC with CSI (Hanly-Tse 98)

- Union of rate regions (polymatroids) achievable by all valid power control policies.

\[
\bigcup \left\{ \mathbf{R} : \sum_{i \in \Gamma} R_i \leq E_h \left[ \frac{1}{2} \log \left( 1 + \sigma^{-2} \sum_{i \in \Gamma} h_i p_i(h) \right) \right], \quad \forall \Gamma \subset \{1, \ldots, K\} \right\}
\]
The power control policy that corresponds to the rate pair \((R_1^*, R_2^*)\) can be found by maximizing \(\mu_1 R_1 + \mu_2 R_2\) subject to the average power constraints, for some \(\mu_1, \mu_2\).

Any \((R_1^*, R_2^*)\) on the curved portion of the boundary is a corner of one of the pentagons.
Optimum Power allocation

- Can be obtained by a greedy algorithm [Hanly-Tse 98], or by using generalized iterative waterfilling [Kaya-Ulukus 2006].
- Has a simultaneous waterfilling nature.
User Cooperation

- Interference is information.
- Some versions of all transmitted signals are received by all nodes.
- User cooperation: exploit overheard information to jointly design encoding, transmit, routing policies.
- Building block towards the analysis of larger networks.

\[
\begin{align*}
Y_0 &= h_{10}X_1 + h_{20}X_2 + n_0 \\
Y_1 &= h_{21}X_2 + n_1 \\
Y_2 &= h_{12}X_1 + n_2
\end{align*}
\]
Motivation
Optimum Power Allocation for the Two User Cooperative MAC

\[ Y_0 = h_{10}X_1 + h_{20}X_2 + n_0 \]
\[ Y_1 = h_{21}X_2 + n_1 \]
\[ Y_2 = h_{12}X_1 + n_2 \]

- Joint work with Sennur Ulukus
MAC with Generalized Feedback

- Gaussian MAC with cooperating encoders [Sendonaris, Erkip, Aazhang]
  - Special case of MAC with generalized feedback [Willems, van der Meulen, Schalkwijk]

- An achievable rate region is obtained by employing
  - Block Markov superposition encoding
    * Inject high rate fresh information to be resolved with the help of upcoming blocks.
    * Send resolution information for previous blocks.
  - Backward decoding
    * After receiving all blocks, decode the resolution information in the last block.
    * Using previously decoded resolution information, sequentially decode earlier blocks.
Gaussian MAC with User Cooperation – No Resource Allocation

Block Markov superposition coding

- Build common information \((X_{12}, X_{21})\)
- Cooperatively send \((U)\)
- Inject new information \((X_{10}, X_{20})\)

\[
X_1 = \sqrt{p_{10}}X_{10} + \sqrt{p_{12}}X_{12} + \sqrt{p_{u1}}U
\]
\[
X_2 = \sqrt{p_{20}}X_{20} + \sqrt{p_{21}}X_{21} + \sqrt{p_{u2}}U
\]

- Amplitude of the each channel’s gain is assumed to be known at the corresponding receiver.
- Phases of all channel gains are assumed known at the receiver and the transmitters
  - Coherent combining.
Gaussian MAC with User Cooperation – Resource Allocation

Block Markov superposition coding

- Build common information \((X_{12}, X_{21})\)
- Cooperatively send \((U)\)
- Inject new information \((X_{10}, X_{20})\)

\[
X_1 = \sqrt{p_{10}(h)}X_{10} + \sqrt{p_{12}(h)}X_{12} + \sqrt{p_{u1}(h)}U
\]
\[
X_2 = \sqrt{p_{20}(h)}X_{20} + \sqrt{p_{21}(h)}X_{21} + \sqrt{p_{u2}(h)}U
\]

- Complete channel state information at the transmitters and the receiver.
- Transmitted codewords can be modulated by channel adaptive power levels
  - Opportunistic cooperation and transmission – use available average power efficiently.
Gaussian MAC with User Cooperation – Resource Allocation

Block Markov superposition coding
- Build common information \( (X_{12}, X_{21}) \)
- Cooperatively send \( (U) \)
- Inject new information \( (X_{10}, X_{20}) \)

\[
X_1 = \sqrt{p_{10}(h)}X_{10} \\
X_2 = \sqrt{p_{20}(h)}X_{20}
\]

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\]

- Complete channel state information at the transmitters and the receiver.
- Transmitted codewords can be modulated by channel adaptive power levels
  - Opportunistic cooperation and transmission – use available average power efficiently.
Achievable Region of Rates with Power Control

• Union over all valid policies $E[p_{i0}(h) + p_{ij}(h) + p_{Ui}(h)] \leq \bar{p}_i$ of pairs \{\(R_1, R_2\)\} that satisfy

$$R_1 < E \left[ \log \left( 1 + \frac{h_{12}p_{12}(h)}{h_{12}p_{10}(h) + \sigma_2^2} \right) + \log \left( 1 + \frac{h_{10}p_{10}(h)}{\sigma_0^2} \right) \right]$$

$$R_2 < E \left[ \log \left( 1 + \frac{h_{21}p_{21}(h)}{h_{21}p_{20}(h) + \sigma_1^2} \right) + \log \left( 1 + \frac{h_{20}p_{20}(h)}{\sigma_0^2} \right) \right]$$

$$R_1 + R_2 < \min \left\{ E \left[ \log \left( 1 + \frac{h_{10}p_{1}(h) + h_{20}p_{2}(h) + 2\sqrt{h_{10}h_{20}p_{U_1}(h)p_{U_2}(h)}}{\sigma_0^2} \right) \right] \right\}$$

$$E \left[ \log \left( 1 + \frac{h_{12}p_{12}(h)}{h_{12}p_{10}(h) + \sigma_2^2} \right) + \log \left( 1 + \frac{h_{21}p_{21}(h)}{h_{21}p_{20}(h) + \sigma_1^2} \right) + \log \left( 1 + \frac{h_{10}p_{10}(h) + h_{20}p_{20}(h)}{\sigma_0^2} \right) \right]$$

• Bounds not concave in power vector $p(h) = [p_{10}(h) \ p_{12}(h) \ p_{U_1}(h) \ p_{20}(h) \ p_{21}(h) \ p_{U_2}(h)]$
Proposition 1. Let the effective channel gains normalized by the noise powers be defined as $s_{ij} = h_{ij}/\sigma_j^2$. Then, for the power control policy $p^*(\mathbf{h})$ that maximizes the sum rate, we need

- $p^*_{10}(\mathbf{h}) = p^*_{20}(\mathbf{h}) = 0$, if $s_{12} > s_{10}$ and $s_{21} > s_{20}$
- $p^*_{10}(\mathbf{h}) = p^*_{21}(\mathbf{h}) = 0$, if $s_{12} > s_{10}$ and $s_{21} \leq s_{20}$
- $p^*_{12}(\mathbf{h}) = p^*_{20}(\mathbf{h}) = 0$, if $s_{12} \leq s_{10}$ and $s_{21} > s_{20}$

\[
\begin{align*}
\left\{ \begin{array}{l}
p^*_{12}(\mathbf{h}) = p^*_2(\mathbf{h}) = 0 \\
\text{OR} \\
\text{OR} \\
p^*_{10}(\mathbf{h}) = p^*_{21}(\mathbf{h}) = 0 \\
\text{if } s_{12} \leq s_{10} \text{ and } s_{21} \leq s_{20}
\end{array} \right. 
\end{align*}
\]
Implications of the Optimal Power Allocation

- Block Markov superposition coding is simpler than originally thought.
  - Each transmitter either sends a cooperation signal or fresh information, but not both!
- The choice at each channel state “only” depends on the channel state.
  - Channel statistics, power constraints play no role in deciding which signals to transmit.
  - Except for the tiny little last case... which usually has very insignificant probability.
- The achievable rate expressions are greatly simplified, and are now concave.
- This simplified coding policy not only maximizes the sum rate, but also the individual rate constrains on $R_1$ and $R_2$, and is optimal in terms of the entire rate region.
- Concave optimization problem over a convex constraint set, but non-differentiable.
Simplified Rate Region – Example

• Assume $s_{12} > s_{10}$, $s_{21} > s_{20}$ to illustrate the simplified rate region.

$$R_1 < E \left[ \log \left( 1 + s_{12} p_{12}(h) \right) \right]$$
$$R_2 < E \left[ \log \left( 1 + s_{21} p_{21}(h) \right) \right]$$

$$R_1 + R_2 < \min \left\{ E \left[ \log \left( 1 + s_{10} p_{1}(h) + s_{20} p_{2}(h) + 2 \sqrt{s_{10} s_{20} p_u(h) p_u(h)} \right) \right], 
E \left[ \log \left( 1 + s_{12} p_{12}(h) \right) + \log \left( 1 + s_{21} p_{21}(h) \right) \right] \right\}$$

• Inequalities define either a pentagon like in the traditional MAC, or a triangle.

• All bounds concave in powers, and so is any weighted sum $\mu_1 R_1 + \mu_2 R_2$ at the corners.

• Sum rate not differentiable where the arguments of the min are equal.
Rate Maximization Using Subgradient Method

- Points on the rate region boundary can be obtained by maximizing $C_\mu = \mu_1 R_1 + \mu_2 R_2$.

- The optimization problem for arbitrary priorities $\mu_1$ and $\mu_2$ is given by

$$\max_{p(h)} \mu_1 R_1 + \mu_2 R_2$$

subject to

$$E_{3,4} [p_{10}(h)] + E_{1,2} [p_{12}(h)] + E [p_{U1}(h)] \leq \bar{p}_1$$

$$E_{2,4} [p_{20}(h)] + E_{1,3} [p_{21}(h)] + E [p_{U2}(h)] \leq \bar{p}_2$$

- $\{R_1, R_2\}$ is the corner of the pentagon obtained for a given power allocation policy.

- Gradient of the objective function does not exist everywhere, find subgradient $g$ instead

$$C_\mu(p') \leq C_\mu(p) + (p' - p)g$$

- Use projected subgradient method to maximize $C_\mu$

$$p(k+1) = [p(k) + \alpha_k g_k]^+$$

- Provably converges for a diminishing stepsize $\alpha_k$ [Shor].
Rate of convergence depends on the stepsize parameter.

Subgradient method need not give a monotonically increasing function value.
Achievable Rate Region for Joint Power Control and User Cooperation

- Optimized power levels enlarge the achievable rate region significantly.
Summary and Conclusions

- Characterized the power control policies that are jointly optimal with Block Markov superposition coding.
- Using sub-gradient methods, obtained optimal power levels and corresponding rate region.
- Joint usage of cooperative diversity and time diversity: major improvements in capacity.
- Encoding and decoding is significantly simplified.
  - Transmitters send either cooperation or fresh information signals, but not both.
- Optimal power policies also dictate MAC and routing policies
  - Cross layer design.
Two User Cooperative OFDMA

- *Joint work with Sezi Bakım
- Divides the entire transmission bandwidth into $N$ orthogonal subchannels.
- Converts a frequency selective fading channel into parallel flat fading subchannels.
- Creates diversity across subchannels.
- Avoids interference, but incurs rate penalty due to orthogonalization of transmissions.
User Cooperation

\begin{align*}
Y_0 &= \sqrt{h_{10}}X_1 + \sqrt{h_{20}}X_2 + Z_0 \\
Y_1 &= \sqrt{h_{21}}X_2 + Z_1 \\
Y_2 &= \sqrt{h_{12}}X_1 + Z_2
\end{align*}

- **Interference** is information.
- Why not take advantage of overheard information in OFDMA?
• Equivalent to $N$ orthogonal cooperative MACs.

• Both users may TX & RX on the same subchannel: makes use of overheard information.

• May cooperate independently over each subchannel (intra-subchannel cooperation),

• May cooperate across subchannels (inter-subchannel cooperation).
Two User Cooperative OFDMA Channel Model

\[ Y_0^{(i)} = \sqrt{h_{10}^{(i)}} X_1^{(i)} + \sqrt{h_{20}^{(i)}} X_2^{(i)} + Z_0^{(i)} \]

\[ Y_1^{(i)} = \sqrt{h_{21}^{(i)}} X_2^{(i)} + Z_1^{(i)} \]

\[ Y_2^{(i)} = \sqrt{h_{12}^{(i)}} X_1^{(i)} + Z_2^{(i)} \]
Scalar MAC – Block Markov Superposition Encoding

- Two user cooperation: each user’s message is divided into two sub-messages
  - \( w_1 = (w_{10}, w_{12}) \), \( w_2 = (w_{20}, w_{21}) \)

- Block Markov superposition coding

<table>
<thead>
<tr>
<th>Purpose</th>
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<tbody>
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<td>Build common information</td>
<td>( X_{kj}(w_{kj}[b], w_{kj}[b-1], \hat{w}_{jk}[b-1]) )</td>
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<td>Cooperatively send</td>
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<tr>
<td>Inject new information</td>
<td>( X_{k0}(w_{k0}[b], w_{kj}[b-1], \hat{w}_{jk}[b-1]) )</td>
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\[
X_1 = \sqrt{p_{10}}X_{10} + \sqrt{p_{12}}X_{12} + \sqrt{p_{U1}}U_1
\]

\[
X_2 = \sqrt{p_{20}}X_{20} + \sqrt{p_{21}}X_{21} + \sqrt{p_{U2}}U_2
\]

\[
p_{k0} + p_{kj} + p_{Uk} = P_k
\]
OFDMA: Intra Subchannel Cooperative Encoding

- Two user cooperation: each user’s message is divided into two sub-messages
  \[ w_1 = (w_{10}, w_{12}), \quad w_2 = (w_{20}, w_{21}) \]

- These two sub-messages are further divided into \( N \) submessages each
  \[ w_{k0} = \{ w_{k0}^{(1)}, \ldots, w_{k0}^{(N)} \}, \quad w_{kj} = \{ w_{kj}^{(1)}, \ldots, w_{kj}^{(N)} \} \]

- Block Markov superposition coding

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\[
X_1^{(i)} = \sqrt{p_{10}^{(i)}}X_{10}^{(i)} + \sqrt{p_{12}^{(i)}}X_{12}^{(i)} + \sqrt{p_{U1}^{(i)}}U_1^{(i)} \\
X_2^{(i)} = \sqrt{p_{20}^{(i)}}X_{20}^{(i)} + \sqrt{p_{21}^{(i)}}X_{21}^{(i)} + \sqrt{p_{U2}^{(i)}}U_2^{(i)} \\
\sum_{i=1}^{N} p_{k0}^{(i)} + p_{kj}^{(i)} + p_{Uk}^{(i)} = P_k
\]
**Intra-Subchannel Cooperative Encoding – Rate Constraints**

- Rate constraints for reliable decoding at users:

\[
R_{12}^{(i)} < E \left[ \log \left( 1 + \frac{s_{12}p_{12}^{(i)}}{s_{12}p_{10}^{(i)} + 1} \right) \right]
\]

\[
R_{21}^{(i)} < E \left[ \log \left( 1 + \frac{s_{21}p_{21}^{(i)}}{s_{21}p_{20}^{(i)} + 1} \right) \right]
\]

- Rate constraints for reliable decoding at receiver:

\[
R_{10}^{(i)} < E \left[ \log \left( 1 + s_{10}p_{10}^{(i)} \right) \right]
\]

\[
R_{20}^{(i)} < E \left[ \log \left( 1 + s_{20}p_{20}^{(i)} \right) \right]
\]

\[
R_{10}^{(i)} + R_{20}^{(i)} < E \left[ \log \left( 1 + s_{10}p_{10}^{(i)} + s_{20}p_{20}^{(i)} \right) \right]
\]

\[
R_{12}^{(i)} + R_{21}^{(i)} + R_{10}^{(i)} + R_{20}^{(i)} < C_s^{(i)} \triangleq E \left[ \log \left( 1 + s_{10}p_1^{(i)} + s_{20}p_2^{(i)} + 2\sqrt{s_{10}s_{20}p_{u1}^{(i)}p_{u2}^{(i)}} \right) \right]
\]
Intra-Subchannel Cooperative Encoding: Issues and Limitations

\[ R_1 < \sum \min \left\{ E \left[ \log \left( 1 + \frac{s_{12}^{(i)} p_{12}^{(i)}}{s_{12}^{(i)} p_{10}^{(i)} + 1} \right) \right] \right\} + E \left[ \log \left( 1 + s_{10}^{(i)} p_{10}^{(i)} \right) \right], C_s^{(i)} \right\} \]

\[ R_2 < \sum \min \left\{ E \left[ \log \left( 1 + \frac{s_{21}^{(i)} p_{21}^{(i)}}{s_{21}^{(i)} p_{20}^{(i)} + 1} \right) \right] \right\} + E \left[ \log \left( 1 + s_{20}^{(i)} p_{20}^{(i)} \right) \right], C_s^{(i)} \right\} \]

\[ R_1 + R_2 < \sum \min \left\{ C_s^{(i)}, E \left[ \log \left( 1 + \frac{s_{12}^{(i)} p_{12}^{(i)}}{s_{12}^{(i)} p_{10}^{(i)} + 1} \right) \right] + E \left[ \log \left( 1 + \frac{s_{21}^{(i)} p_{21}^{(i)}}{s_{21}^{(i)} p_{20}^{(i)} + 1} \right) \right] \right\} \]

\[ + E \left[ \log \left( 1 + s_{10}^{(i)} p_{10}^{(i)} + s_{20}^{(i)} p_{20}^{(i)} \right) \right] \}

- Each submessage is retransmitted over the same subchannel it was received on.
- Does not take advantage of diversity created by OFDMA.
- Rate over each subchannel is limited by the worst link.
Inter-Subchannel Cooperative Encoding

- Can re-partition and re-encode the overall message received over the subchannels:
  - \( w_{kj} \) can be divided into new submessages with rates \( R'_{kj} \)
    * \( w_{12} = \{v_{12}^{(1)}, \ldots, v_{12}^{(N)}\} \), \( w_{21} = \{v_{21}^{(1)}, \ldots, v_{21}^{(N)}\} \)
  - \( \{w_{kj}^{(i)}\}_{i=1}^{N} \) and \( \{v_{kj}^{(i)}\}_{i=1}^{N} \), are just different partitionings of the same message \( w_{kj} \), so their total rates have to be the same:
    \[
    2^{nR_{12}} = 2^{nR_{12}^{(1)} + \ldots + nR_{12}^{(N)}} = 2^{nR'_{12}^{(1)} + \ldots + nR'_{12}^{(N)}},
    \]
    \[
    2^{nR_{21}} = 2^{nR_{21}^{(1)} + \ldots + nR_{21}^{(N)}} = 2^{nR'_{21}^{(1)} + \ldots + nR'_{21}^{(N)}}.
    \]
- Re-encode the new partition onto the subchannels
  - The information received over a subchannel no longer required to be sent over the same subchannel.
Inter-Subchannel Cooperative Encoding – Message Repartitioning

Block (b-1)

User 1

\[ w_{12}^{(1)} \ldots w_{12}^{(i)} \ldots w_{12}^{(N)} \]

\[ \hat{w}_{21}^{(1)} \ldots \hat{w}_{21}^{(i)} \ldots \hat{w}_{21}^{(N)} \]

User 2

\[ w_{21}^{(1)} \ldots w_{21}^{(i)} \ldots w_{21}^{(N)} \]

\[ \hat{w}_{12}^{(1)} \ldots \hat{w}_{12}^{(i)} \ldots \hat{w}_{12}^{(N)} \]

Block (b)

User 1

\[ v_{12}^{(l)} \hat{v}_{21}^{(l)} \]

\[ \ldots \]

\[ \hat{v}_{12}^{(i)} \hat{v}_{21}^{(i)} \]

\[ \ldots \]

\[ v_{12}^{(N)} \hat{v}_{21}^{(N)} \]

Receiver

\[ \tilde{v}_{12}^{(l)} \tilde{v}_{21}^{(l)} \]

\[ \ldots \]

\[ \tilde{v}_{12}^{(i)} \tilde{v}_{21}^{(i)} \]

\[ \ldots \]

\[ \tilde{v}_{12}^{(N)} \tilde{v}_{21}^{(N)} \]

User 2

\[ w_{12}^{(1)} \]

\[ \hat{w}_{12} \]

\[ \tilde{w}_{12} \]

\[ \tilde{w}_{21} \]
Inter-Subchannel Cooperation: Encoding and Decoding

- **Encoding**

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<tr>
<td>Build common information</td>
<td>$X^{(i)}<em>{k,j} \left( w^{(i)}</em>{k,j}[b], v^{(i)}<em>{k,j}[b - 1], \hat{v}^{(i)}</em>{j,k}[b - 1] \right)$</td>
</tr>
<tr>
<td>Cooperatively send</td>
<td>$U^{(i)}<em>{k} \left( v^{(i)}</em>{k,j}[b - 1], \hat{v}^{(i)}_{j,k}[b - 1] \right)$</td>
</tr>
<tr>
<td>Inject new information</td>
<td>$X^{(i)}<em>{k,0} \left( w^{(i)}</em>{k,0}[b], v^{(i)}<em>{k,j}[b - 1], \hat{v}^{(i)}</em>{j,k}[b - 1] \right)$</td>
</tr>
</tbody>
</table>

- **Decoding**

  - Each user uses joint typicality check at the end of each block.
  
  - Receiver uses backwards decoding to determine the transmitted messages
    
    - For each subchannel determine $\hat{v}^{(i)}_{21}[b - 1], \hat{v}^{(i)}_{12}[b - 1], \hat{w}^{(i)}_{10}[b]$ and $\hat{w}^{(i)}_{20}[b]$
    
    - Estimates of the re-partitioned cooperative messages $\hat{v}^{(i)}_{k,j}[b - 1]$ are converted to estimates of the cooperative messages $\hat{w}^{(i)}_{k,j}[b - 1]$ via match-up table available at the users and the receiver.
Inter-Subchannel Cooperative Encoding – Rate Constraints

- Rate constraints for reliable decoding at users:
  \[ R_{12}^{(i)} < E \left[ \log \left( 1 + \frac{s_{12}^{(i)}}{s_{12}^{(i)}p_{10} + 1} \right) \right] \]
  \[ R_{21}^{(i)} < E \left[ \log \left( 1 + \frac{s_{21}^{(i)}}{s_{21}^{(i)}p_{20} + 1} \right) \right] \]

- Rate constraints for reliable decoding at receiver:
  \[ R_{10}^{(i)} < E \left[ \log \left( 1 + s_{10}^{(i)}p_{10} \right) \right] \]
  \[ R_{20}^{(i)} < E \left[ \log \left( 1 + s_{20}^{(i)}p_{20} \right) \right] \]
  \[ R_{10}^{(i)} + R_{20}^{(i)} < E \left[ \log \left( 1 + s_{10}^{(i)}p_{10} + s_{20}^{(i)}p_{20} \right) \right] \]
  \[ R_{12}^{(i)} + R_{21}^{(i)} + R_{10}^{(i)} + R_{20}^{(i)} < C_s^{(i)} \triangleq E \left[ \log \left( 1 + s_{10}^{(i)}p_{1}^{(i)} + s_{20}^{(i)}p_{2}^{(i)} + 2\sqrt{s_{10}^{(i)}s_{20}^{(i)}}p_{u1}^{(i)}p_{u2}^{(i)} \right) \right] \]
Inter-Subchannel Cooperative Encoding - Achievable Rate Region

- Achievable rate region is equivalent to the closure of the convex hull of all rate pairs:

\[
R_1 < \sum_i E \left[ \log \left( 1 + \frac{s_{12}(i)p_{12}}{s_{12}(i)p_{10} + 1} \right) \right] + E \left[ \log \left( 1 + s_{10}(i)p_{10} \right) \right]
\]

\[
R_2 < \sum_i E \left[ \log \left( 1 + \frac{s_{21}(i)p_{21}}{s_{21}(i)p_{20} + 1} \right) \right] + E \left[ \log \left( 1 + s_{20}(i)p_{20} \right) \right]
\]

\[
R_1 + R_2 < \min \left\{ \sum_i C_s(i), \sum_i E \left[ \log \left( 1 + \frac{s_{12}(i)p_{12}}{s_{12}(i)p_{10} + 1} \right) \right] + E \left[ \log \left( 1 + \frac{s_{21}(i)p_{21}}{s_{21}(i)p_{20} + 1} \right) \right]
\]

\[
+ E \left[ \log \left( 1 + s_{10}(i)p_{10} + s_{20}(i)p_{20} \right) \right] \right\}
\]
Achievable Rates for Two User Cooperative OFDMA

Achievable Rate Regions for Cooperative vs Non-cooperative OFDMA

- $R_1$
- $R_2$

For cooperative OFDMA:
- $E[h_{10}^{(1)}] = E[h_{20}^{(1)}] = 0.6$,
- $E[h_{12}^{(1)}] = E[h_{21}^{(1)}] = 0.9$,
- $E[h_{10}^{(2)}] = E[h_{20}^{(2)}] = 0.6$,
- $E[h_{12}^{(2)}] = E[h_{21}^{(2)}] = 0.9$,
- $E[h_{10}^{(3)}] = E[h_{20}^{(3)}] = 0.8$,
- $E[h_{12}^{(3)}] = E[h_{21}^{(3)}] = 0.6$.

For non-cooperative OFDMA:
- $E[h_{10}] = E[h_{20}] = 0.8$,
- $E[h_{12}] = E[h_{21}] = 0.6$.

Legend:
- Red solid line: inter-subchannel coop
- Blue dotted line: intra-subchannel coop
- Dashed line: no cooperation
Achievable Rates for Two User Cooperative OFDMA

Achievable Rate Regions for Cooperative vs Non–cooperative OFDMA

- inter–subchannel coop
- intra–subchannel coop
- no cooperation

E[h_{10}^{(1)}]=E[h_{20}^{(2)}]=0.3,
E[h_{10}^{(1)}]=E[h_{20}^{(2)}]=0.6,
E[h_{10}^{(3)}]=E[h_{20}^{(3)}]=0.6,
E[h_{12}^{(1)}]=E[h_{21}^{(1)}]=0.9,
E[h_{12}^{(2)}]=E[h_{21}^{(2)}]=0.9,
E[h_{12}^{(3)}]=E[h_{21}^{(3)}]=0.4.
Summary and Conclusions

- Introduced a two user cooperative OFDMA system, and proposed two encoding strategies based on block Markov superposition encoding:
  - Intra-subchannel cooperative encoding
  - Inter-subchannel cooperative encoding
- Derived rate region expressions and obtained the achievable rate regions for both encoding strategies
- Showed that re-partitioning and re-encoding of the cooperative messages across subchannels;
  - Always superior to intra-subchannel cooperative encoding.
  - Significant improvement with respect to non-cooperative OFDMA.
- Can we do any better? Yes! Power control.
Power control for cooperative OFDMA

- The structure of the problem is very similar to the scalar case.
  - Now, we have an additional sum constraint for powers, over the sub-channels.
- The dimensionality of the problem is $N$ times the scalar case.
- Can still use subgradients. A little slow, but it works.
- Can also exploit the convex nature of the problem, if we formulate it correctly.
Differentiable Reformulation of Sum-Rate Maximization Problem

- Idea: get rid of the minimum operation:

\[
\max_{p^{(i)}(s)} r
\]

s.t. \[
\begin{align*}
    r &\leq \sum_i E \left[ \log \left( 1 + s_{10}^{(i)}(p_{12}(s) + p_{U_1}(s)) + s_{20}^{(i)}(p_{21}(s) + p_{U_2}(s)) \right) \\
    &\quad + 2\sqrt{s_{10}^{(i)}s_{20}^{(i)}p_{U_1}(s)p_{U_2}(s)} \right] \\
    r &\leq \sum_i E \left[ \log(1 + p_{12}(s)s_{12}^{(i)}) + \log(1 + p_{21}(s)s_{21}^{(i)}) \right] \\
\sum_i E \left[ p_{12}(s) \right] + E \left[ p_{U_1}(s) \right] &\leq \bar{p}_1 \\
\sum_i E \left[ p_{21}(s) \right] + E \left[ p_{U_2}(s) \right] &\leq \bar{p}_2 \\
p_{12}(s), p_{U_1}(s), p_{21}(s), p_{U_2}(s) &\geq 0, \quad \forall s
\end{align*}
\]
Lagrangian Approach

\[ L = r + \gamma_1 \left( \sum_i \left( E \left[ \log (1 + s_{12}^{(i)} p_{12}^{(i)}(s)) + \log (1 + s_{21}^{(i)} p_{21}^{(i)}(s)) \right] \right) - r \right) \\
+ \gamma_2 \left( \sum_i E \left[ \log \left( 1 + s_{10}^{(i)} (p_{12}^{(i)}(s) + p_{U_1}^{(i)}(s)) + s_{20}^{(i)} (p_{21}^{(i)}(s) + p_{U_2}^{(i)}(s)) \right) \\
+ 2 \sqrt{s_{10}^{(i)} s_{20}^{(i)} p_{U_1}^{(i)}(s) p_{U_2}^{(i)}(s)} \right] - r \right) + \lambda_1 \left( \bar{p}_1 - \sum_i \left( E \left[ p_{12}^{(i)}(s) + p_{U_1}^{(i)}(s) \right] \right) \right) \\
+ \lambda_2 \left( \bar{p}_2 - \sum_i \left( E \left[ p_{21}^{(i)}(s) + p_{U_2}^{(i)}(s) \right] \right) \right) + \varepsilon_1^{(i)}(s) p_{12}^{(i)}(s) + \varepsilon_2^{(i)}(s) p_{U_1}^{(i)}(s) \\
+ \varepsilon_3^{(i)}(s) p_{21}^{(i)}(s) + \varepsilon_4^{(i)}(s) p_{U_2}^{(i)}(s) \right). \]
Karush-Kuhn-Tucker Conditions

\[ \gamma_1 \frac{s_{12}^{(i)}}{1 + s_{12}^{(i)} p_{12}(s)} + \gamma_2 \frac{s_{10}^{(i)}}{D^{(i)}} \leq \lambda_1 \]

\[ \gamma_1 \frac{s_{21}^{(i)}}{1 + s_{21}^{(i)} p_{21}(s)} + \gamma_2 \frac{s_{20}^{(i)}}{D^{(i)}} \leq \lambda_2 \]

\[ \gamma_2 \frac{\sqrt{s_{10}^{(i)} s_{20}^{(i)} p_{U_2}(s) + s_{10}^{(i)} p_{U_1}(s)}}{D^{(i)} \sqrt{p_{U_1}(s)}} \leq \lambda_1 \]

\[ \gamma_2 \frac{\sqrt{s_{10}^{(i)} s_{20}^{(i)} p_{U_1}(s) + s_{20}^{(i)} p_{U_2}(s)}}{D^{(i)} \sqrt{p_{U_2}(s)}} \leq \lambda_2 \]

- \( \gamma_1 + \gamma_2 = 1 \)
- Each condition satisfied with strict equality, if the corresponding power is positive.
- All we need to do is find \( \lambda_i \) and \( \gamma_1 \)
Structure of Optimal Power Allocation

• When $p_{U_1}$ and $p_{U_2}$ are both positive,

$$p_{12}^{(i)}(s) = \left( \frac{\gamma_1 \left( \lambda_2 s_{10}^{(i)} + \lambda_1 s_{20}^{(i)} \right) - 1}{\lambda_1 s_{20}^{(i)}} \right)^+$$

$$p_{21}^{(i)}(s) = \left( \frac{\gamma_1 \left( \lambda_2 s_{10}^{(i)} + \lambda_1 s_{20}^{(i)} \right) - 1}{\lambda_2 s_{10}^{(i)}} \right)^+$$

• When both are zero, $p_{12}$ and $p_{21}$ solved from,

$$\frac{\gamma_1}{1 + s_{12}^{(i)} p_{12}(s)} + \frac{\gamma_2}{1 + s_{10}^{(i)} p_{12}(s) + s_{20}^{(i)} p_{21}(s)} \leq \lambda_1$$

$$\frac{\gamma_1}{1 + s_{21}^{(i)} p_{21}(s)} + \frac{\gamma_2}{1 + s_{10}^{(i)} p_{21}(s) + s_{20}^{(i)} p_{21}(s)} \leq \lambda_2$$

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Iterative Power Allocation Algorithm

- All powers can be computed using KKT conditions, by iteratively searching for Lagrange multipliers.
- Not exactly closed form: $p_{U_1}$ and $p_{U_2}$’s depend on $p_{12}$ and $p_{21}$, and vice versa.
- Objective function concave, constraints strictly convex, Cartesian nature across users:
  - Can solve the users’ powers iteratively – one user at a time.
  - Start by assuming $p_U$’s positive, and iterate. Converges to optimum.
Optimal Power Allocation over Fading States– U-D links high

(a) Power levels, $p_{12}^{(1)}$  
(b) Power level, $p_{21}^{(1)}$

(c) Power level, $p_{U1}^{(1)}$  
(d) Power level, $p_{U2}^{(1)}$
Optimal Power Allocation over Fading States– U-D links moderate
Cooperative OFDMA – Achievable Rates with Power Control

Achievable Rate Regions for Cooperative vs Non-Coop OFDMA w/o Power Control

- intra-subchannel coop, no power contr
- inter-subchannel coop, no power contr
- no cooperation, no power contr
- half-duplex coop, no power contr
- inter-subchannel coop, power contr

\[ E[h_{10}^{(1)}] = E[h_{20}^{(2)}] = 0.3, \]
\[ E[h_{12}^{(1)}] = E[h_{21}^{(2)}] = 0.6, \]
\[ E[h_{10}^{(3)}] = E[h_{20}^{(3)}] = 0.6, \]
\[ E[h_{12}^{(3)}] = E[h_{21}^{(3)}] = 0.4. \]