Achieving the Capacity Region Boundary of Fading CDMA Channels via Generalized Iterative Waterfilling*

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Abstract

We characterize the optimum power control policies that achieve arbitrary rate tuples on the boundary of the capacity region of a power controlled, code division multiple access (CDMA) system in a fading channel with perfect channel state information. We propose a "generalized" waterfilling approach, and provide an iterative algorithm that solves for the optimum power allocation policy, for a given arbitrary rate tuple on the boundary of the capacity region. We then investigate the effects of limited feedback on the capacity region, and demonstrate that a good power control policy may require only a very low rate feedback.

Index terms: CDMA, fading channels, capacity region, power control, generalized iterative waterfilling, limited feedback.

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1 Introduction

The capacity limits of communication systems subject to fading have recently drawn significant attention, and in the last decade, several results regarding the information theoretic capacities of many channel models have been reported. The particularly interesting types of channel models are those where the transmitter(s) and receiver(s) are able to track the variations in the channel, and therefore are capable of allocating the system resources and adapting their coding and decoding strategies to the variations in the channel, in order to improve the capacity. In this paper, we consider the uplink of a fading CDMA channel, and the resources we allocate are the available transmit powers.

The problem of power allocation in order to maximize the information theoretic capacity in the presence of fading was first studied for a single user channel in [2], where it was shown that, subject to an average power constraint and under the ergodicity assumption on the fading process, the ergodic capacity of the channel is maximized by allocating the total power of the user according to a waterfilling strategy, where the user "waterfills" its power in time, over the inverse of the channel states.

For multiple access channels, the capacity region is defined as the set of achievable rate tuples. For a scalar multiple access channel (MAC), [3] solved the power allocation problem with the goal of achieving a special rate tuple on the capacity region, the one that achieves the ergodic *sum* capacity. There, it was shown that in order to achieve the sum capacity, only the strongest user may transmit at any given time, and the optimun power control policy is again waterfilling, over disjoint sets of channel states.

The entire capacity region, and the corresponding power control policies for the scalar MAC were characterized in [4]. The capacity region is shown to be a union of the capacity regions (polymatroids) achievable by all valid power allocation policies, i.e., the policies that satisfy the average power constraints. The optimal power allocation policy for each rate tuple on the capacity region is obtained by a greedy algorithm, which compares certain marginal utility functions, and makes use of the generalized symmetry properties of the rank function of the polymatroid corresponding to the rate tuple in question.

The capacity region for a non-fading vector MAC, where the total average powers of the components of the transmitted vectors are constrained, is given by [5]. There, also an iterative waterfilling algorithm which allocates the powers over the components of the transmitted vector in order to maximize the *sum* capacity was proposed. The power allocation problem for a fading vector MAC was considered in [6], again with the aim of maximizing the *sum* capacity. It was shown that, the optimal power allocation in the fading case as well satisfies the Karush-Kuhn-Tucker (KKT) conditions, which can also be interpreted as simultaneous waterfilling, where the water levels are matrices.

In [7], the capacity region of a power controlled fading CDMA channel with perfect channel state information at the transmitters and the receiver was obtained. Also, the power allocation policy that achieves the *sum* capacity point on the capacity region boundary was found, and it was shown that the optimal power allocation policy is a simultaneous waterfilling of powers over the inverse of the SIRs the users would obtain at the output of MMSE receivers if they transmitted with unit powers. It was also shown that, similar to [6], a one-user-at-a-time iterative waterfilling algorithm can be used to solve these simultaneous waterfilling conditions, and therefore to obtain the optimal power distributions of all users over all fading states.

In this paper, we consider the problem of solving for the power allocation policy that achieves an *arbitrary rate tuple* on the capacity region of fading CDMA. As in [4] and [5], this problem is equivalent to a maximization of a weighted sum of rates, subject to average power constraints. However, the algorithm proposed in [4] to find the power allocation policies that achieve the boundary points of the scalar MAC does not generalize to the CDMA case. This is due to the fact that the generalized symmetry properties of the rank functions that describe the capacity region of a scalar MAC does not carry over to an arbitrary CDMA system, in which non-identical signature sequences are employed. Instead, we make use of the concavity of the objective function and the convexity of the constraints, and write the KKT conditions at each fading state, for a given set of weights. We then develop a "generalized" waterfilling approach, where we gradually pour some power at some or all channel states until all the KKT conditions are satisfied. Using this approach, we propose a one-user-at-a-time algorithm which is similar in spirit to those in [5,7,8], and show that it converges to the optimum power allocation for any given point on the boundary of the capacity region. This algorithm, while providing a systematic solution to the capacity achieving power allocation problem in fading CDMA, also provides as a special case, an intuitive approach to the power allocation for scalar MAC in [4].

We also relax the somewhat impractical assumption of perfect channel state information at the transmitters, and investigate the effects of limited feedback rates from the receiver to the transmitters. We show that even with very low feedback rates, it is possible to achieve rates very close to the capacity region boundary.

2 Problem Definition

We consider a symbol synchronous CDMA system with processing gain N, where all K users transmit to a single receiver site. In the presence of fading and AWGN, the received signal is given by [9],

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i h_i} x_i \mathbf{s}_i + \mathbf{n} \tag{1}$$

where, for user i, x_i denotes the information symbol with $\mathrm{E}[x_i^2] = 1$, $\mathbf{s}_i = [s_{i1}, \cdots, s_{iN}]^\top$ denotes the unit energy signature sequence, $\sqrt{h_i}$ denotes the random and continuously distributed channel gain, and p_i denotes the transmit power; $\mathbf{n}$ is a zero-mean Gaussian random vector with covariance $\sigma^2\mathbf{I}_N$. We assume that the receiver and all of the transmitters have perfect knowledge of the channel states of all users represented as a vector $\mathbf{h} = [h_1, \cdots, h_K]^\top$, and the components of $\mathbf{h}$ are independent. We further assume that although the fading is slow enough to ensure constant channel gain in a symbol interval, it is fast enough so that within the transmission time of a block of symbols the long term ergodic properties of the fading process can be observed [10].

For the CDMA system given by (1), let the transmitters be able to choose their powers as a function of the channel states, subject to the average power constraints $E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i$. The following theorem from [7] gives the set of long term achievable rates, i.e., the capacity region, for fading CDMA.

Theorem 1 ([7]) *The capacity region of a fading CDMA channel under additive white Gaussian noise, where users have perfect channel state information and allocate their powers as a function of the CSI subject to average power constraints $E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i$ is given by,*

$$\bigcup_{\{\mathbf{p}(\mathbf{h}):\, E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i,\, \forall i\}} \left\{ \mathbf{R} : \sum_{i\in\Gamma} R_i \le E_{\mathbf{h}} \left[\frac{1}{2} \log \left| \mathbf{I}_N + \sigma^{-2} \sum_{i\in\Gamma} h_i p_i(\mathbf{h}) \mathbf{s}_i \mathbf{s}_i^\top \right| \right], \forall \Gamma \subset \{1, \cdots, K\} \right\} \tag{2}$$

Figure 1 illustrates a typical capacity region for some fixed signature sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ in a two user setting. Each of the pentagons corresponds to a valid power allocation policy. We have shown in [7] that the capacity region for fading CDMA is in general not strictly convex, and there may be a flat portion on the boundary of the capacity region, which coincides with the dominant face of the rate region corresponding to the sum capacity

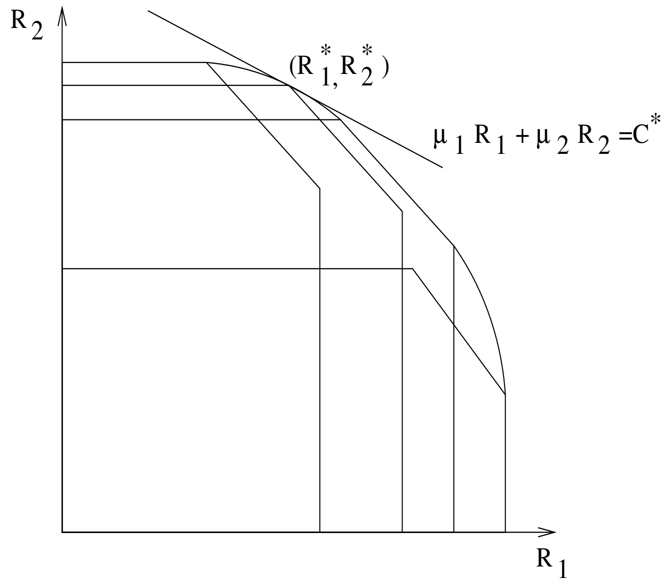


Figure 1: Sample two user capacity region.

maximizing power control policy. Now, note that, for any given pair of non-negative numbers μ_1 and μ_2, there exists a point (or there exist points) (R_1^*, R_2^*) on the boundary of the capacity region, such that the line $\mu_1 R_1 + \mu_2 R_2 = C$ is tangent to the capacity region for some $C = C^*(\mu_1, \mu_2)$, and in fact $C^*(\mu_1, \mu_2)$ is the maximum achievable value of $\mu_1 R_1 + \mu_2 R_2$. Therefore, the problem of finding the power control policy that corresponds to the rate pair (R_1^*, R_2^*) is equivalent to maximizing $\mu_1 R_1 + \mu_2 R_2$ subject to the average power constraints. Here, μ_is can be interpreted as the priorities assigned to each user. The boundary of the capacity region can be traced by varying these priorities μ_i. The desired rate pair (R_1^*, R_2^*) is either the corner of one of the pentagons specified by a power allocation policy as in (2), or it lies on one of the flat portions. If it is a corner, its coordinates can be written as a function of the power allocation policy using (2), and the maximization can be carried out. The case where (R_1^*, R_2^*) lies on one of the flat portions correspond to either the rather easier case where we want to maximize the sum capacity, which is solved in [7, 8], or the trivial case where one of the μ_is is zero, and the problem reduces to a single user problem.

Having introduced the reasoning in the simple two user case, we now define our problem in the general K user case. Without loss of generality, assume $\mu_K > \cdots > \mu_1$. Then, the optimum power allocation policy for $\{\mu_i\}_{i=1}^K$ is the solution to the maximization problem,

$$\begin{aligned}
\max_{\mathbf{p}(\mathbf{h})} \quad & \frac{1}{2} E_{\mathbf{h}} \left[\mu_1 \log \left| \mathbf{I}_N + \sigma^{-2} \mathbf{S} \mathbf{D}(\mathbf{h}) \mathbf{S}^\top \right| + \sum_{i=2}^{K} (\mu_i - \mu_{i-1}) \log \left| \mathbf{I}_N + \sigma^{-2} \mathbf{S}_{E_i} \mathbf{D}_{E_i}(\mathbf{h}) \mathbf{S}_{E_i}^\top \right| \right] \\
\text{s.t.} \quad & E_{\mathbf{h}}[p_i(\mathbf{h})] \leq \bar{p}_i, \qquad i = 1, \cdots, K \\
& p_i(\mathbf{h}) \geq 0, \quad \forall\ \mathbf{h}, \quad i = 1, \cdots, K
\end{aligned} \tag{3}$$

where $\mathbf{S} = [\mathbf{s}_1 \cdots \mathbf{s}_K]$, $\mathbf{D}(\mathbf{h}) = \text{diag}[p_1(\mathbf{h})h_1, \cdots, p_K(\mathbf{h})h_K]$, $E_i = \{i, \cdots, K\}$ and $\mathbf{p}(\mathbf{h}) = [p_1(\mathbf{h}), \cdots p_K(\mathbf{h})]$. Here, $\mathbf{D}_{E_i}$ and $\mathbf{S}_{E_i}$ refer to sub-matrices containing only the received powers and signature sequences of the users in the subset E_i. Note that, this is the fading CDMA version of equation (3) in [5], and is similar to equation (17) for the scalar case in [4].

3 Generalized Iterative Waterfilling

Let us denote the objective function in (3) by $C_{\boldsymbol{\mu}}(p_1(\mathbf{h}), \cdots, p_K(\mathbf{h}))$, where $\boldsymbol{\mu} = [\mu_1, \cdots, \mu_K]$. In order to solve (3), we first note that the objective function is concave in the power vector $\mathbf{p}(\mathbf{h})$, and further, it is strictly concave in the individual components $p_i(\mathbf{h})$ of $\mathbf{p}(\mathbf{h})$. The constraint set is convex (in fact, affine). Therefore, the unique global solution to the maximization problem in (3) should satisfy the extended KKT conditions, which can be shown to reduce to,

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + p_k(\mathbf{h})} \leq \lambda_k, \ \forall\ \mathbf{h},\ k = 1, \cdots, K \tag{4}$$

where, we have defined $\mu_0 \triangleq 0$ for notational convenience. Here, $a_{ki}(\mathbf{h})$ for $i \leq k \leq K$ is given by,

$$a_{ki}(\mathbf{h}) = \frac{1}{\sigma^{-2} h_k \mathbf{s}_k^\top \left(\mathbf{I}_N + \sigma^{-2} \sum_{j=i, j\neq k}^{K} p_j(\mathbf{h}) h_j \mathbf{s}_j \mathbf{s}_j^\top \right)^{-1} \mathbf{s}_k} \tag{5}$$

Note that, this quantity can be identified as the inverse of the SIR user k would obtain at the output of an MMSE filter if it transmitted with unit power, when users $i, i+1, \cdots, K$ are active. The condition in (4) is satisfied with equality at some $\mathbf{h}$, if $p_k(\mathbf{h}) > 0$. Since the optimum power allocation policy for a given $\boldsymbol{\mu}$ should simultaneously satisfy all the conditions given by (4), and the optimum power of each user k at each fading state $\mathbf{h}$ depends on the power allocations of all other users at that state through $a_{ki}(\mathbf{h})$, it is hard to analytically solve for the optimum policy from the KKT conditions. Therefore, to proceed, we devise an iterative algorithm. Consider optimizing the power of *only* user k over all channel states, given the powers of all other users at all channel states,

$$\begin{aligned} p_k^{n+1} &= \arg\max_{p_k} \; C_{\boldsymbol{\mu}} \left(p_1^{n+1}, \cdots, p_{k-1}^{n+1}, p_k, p_{k+1}^{n} \cdots, p_K^{n} \right) \\ &= \arg\max_{p_k} \; C_{\boldsymbol{\mu}}^{k} \left(p_k \right) \end{aligned} \tag{6}$$

where $C_{\boldsymbol{\mu}}^{k}(p_k)$ denotes the first k terms in (3), i.e., $i = 1, \cdots, k$, that contain contributions from user k to $C_{\boldsymbol{\mu}}(\mathbf{p}(\mathbf{h}))$.

The convergence of such an algorithm has been proved for the case of sum capacity in [7,8] for fading channels, and in [5] for non-fading channels. The objective function here satisfies the same concavity and strict concavity properties as the sum capacity, i.e., it is concave in the power vector $\mathbf{p}(\mathbf{h})$ and strictly concave in its individual components $p_k(\mathbf{h})$, and the constraint set is the same as in [7, 8]. Therefore, the proof in [7, 8] immediately applies to the case of unequal μ_is here, and the update (6) converges to the optimal power allocation by [11, Prop. 3.9]. Thus, it is sufficient to consider separately finding the solution $p_k(\mathbf{h})$ that satisfies the kth KKT condition in (4) for each user k, while keeping the powers of all other users $j \neq k$ as fixed and known quantities.

Let us concentrate on user k, and fix $p_j(\mathbf{h})$, $j \neq k$. It can be shown that, the solution to (6) subject to the average power constraint on $p_k(\mathbf{h})$ should satisfy the KKT condition for the single user problem,

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + p_k(\mathbf{h})} \leq \tilde{\lambda}_k, \quad \forall\, \mathbf{h} \tag{7}$$

We note here that $\tilde{\lambda}_k$ is in general different from the Lagrange multiplier λ_k in (4), since the powers we have fixed for the other users need not be the optimal powers. Eventually, since the iterative algorithm converges to the optimal powers, we know that $\tilde{\lambda}_k$ will converge to λ_k.

We will next argue how this condition can be interpreted as a "generalized" waterfilling. First assume no power has yet been allocated to any channel state. Define the inverse of the left hand side of (7) evaluated at $p_k(\mathbf{h}) = 0$ for all $\mathbf{h}$ by,

$$b_k(\mathbf{h}) = \left(\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h})} \right)^{-1} \tag{8}$$

Then, sort $b_k(\mathbf{h})$ over all channel states $\mathbf{h}$ in increasing order. Since user k has to satisfy its average power constraint, it has to put some power to a non-zero probability subset, say Ω, of all possible channel states. At the channel states where user k transmits with positive power, (7) needs to be satisfied with equality. Let user k start pouring some of its available power to the state which gives the lowest $b_k(\mathbf{h})$, say $\mathbf{h}'$. Next, pick another state $\mathbf{h}''$, such that $b_k(\mathbf{h}') < b_k(\mathbf{h}'')$. User k starts transmitting at $\mathbf{h}''$ only if (i) it has already poured some powers $q_k(\mathbf{h})$ to all states $\mathbf{h}$ such that $b_k(\mathbf{h}) < b_k(\mathbf{h}'')$, (ii) it still has some power left to allocate, and (iii) the already allocated powers satisfy

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + q_k(\mathbf{h})} = b_k^{-1}(\mathbf{h}''), \quad \forall\, \mathbf{h} : b_k(\mathbf{h}) \leq b_k(\mathbf{h}'') \tag{9}$$

Before going any further, using the current construction, let us revisit the sum capacity case in [7,8] where μ_i, $i = 1, \cdots, K$, are all equal to 1. In this case, from (8), $b_k(\mathbf{h}) = a_{k1}(\mathbf{h})$, and it can be easily seen that the described procedure produces the ordinary waterfilling solution; user k will pour its power over $a_{k1}(\mathbf{h}) = b_k(\mathbf{h})$, until all the available power is used. The optimal power value at $\mathbf{h}$ is the difference between the water level $1/\tilde{\lambda}_k$ and the base level $b_k(\mathbf{h})$, whenever the difference is positive; it is zero otherwise, i.e.,

$$p_k(\mathbf{h}) = \left(\frac{1}{\tilde{\lambda}_k} - b_k(\mathbf{h}) \right)^+ \tag{10}$$

The main subtlety in solving for the optimal powers in the arbitrary μ_is case is that, there are more than one terms that involve $p_k(\mathbf{h})$ on the left hand side of (7), and thus the optimal $p_k(\mathbf{h})$ is no longer given by a nice expression such as (10), but is rather the solution to a polynomial equation, obtained by equating the denominators in (7). Therefore, the optimal power levels lose their traditional waterfilling interpretation. However, we can still see the procedure described here as a type of waterfilling, as it gradually equalizes the base levels $b_k(\mathbf{h})$, and solves for the power levels required for such equalization, hence the name "generalized" waterfilling.

Generalized waterfilling yields the optimum power allocation because of the fact that by construction, the KKT conditions are satisfied when all average power is used. To see this, let us denote the left hand side of (9) by $L(\mathbf{h}, q_k(\mathbf{h}))$. We keep increasing $q_k(\mathbf{h}'')$ on the left hand side of (9) gradually. Letting $p_k(\mathbf{h}) = q_k(\mathbf{h})$ when the solution $q_k(\mathbf{h})$ obtained from (9) satisfies the average power constraint, and taking $\tilde{\lambda}_k \triangleq L(\mathbf{h}, p_k(\mathbf{h}))$, we see that the solution

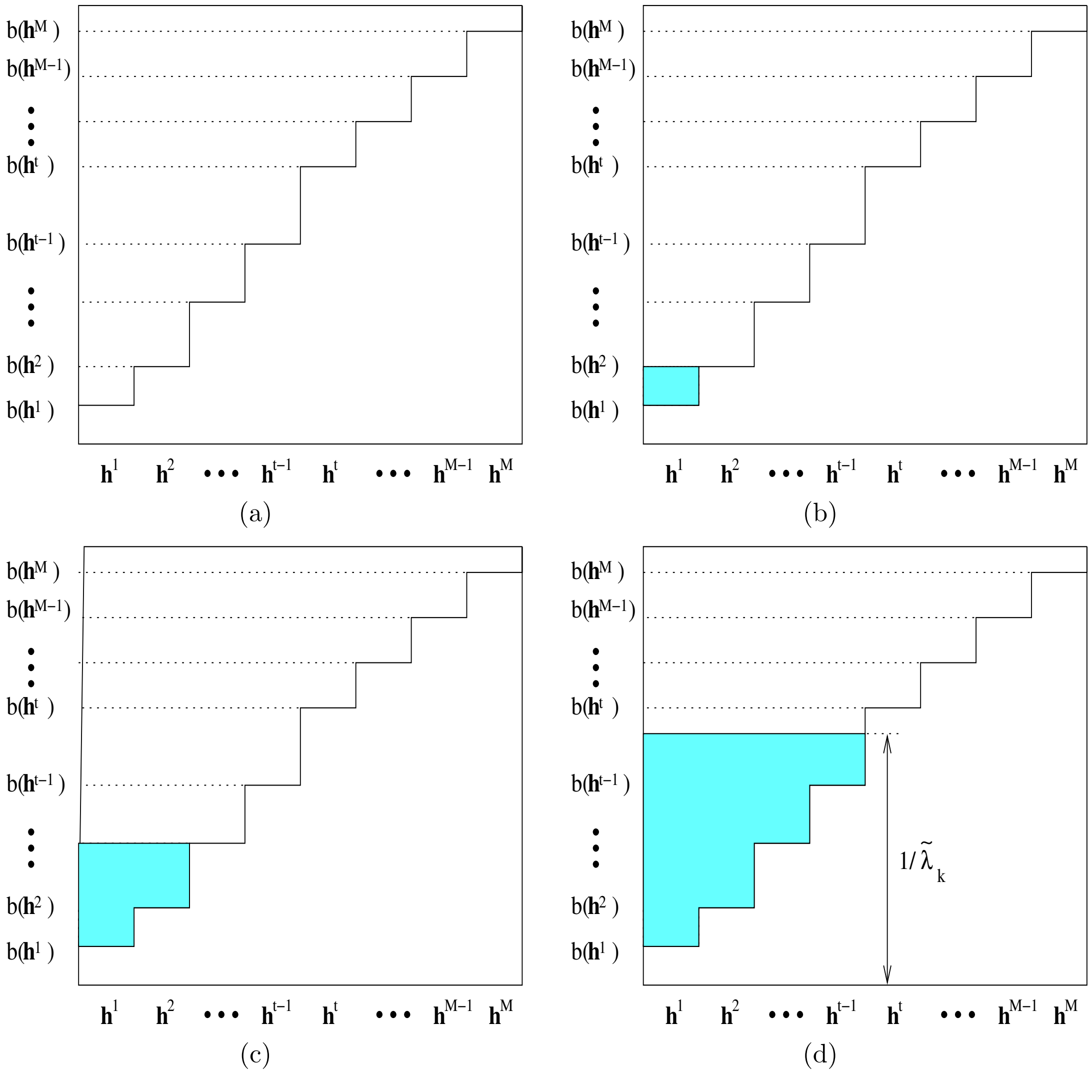


Figure 2: Illustration of the generalized waterfilling.

$p_k(\mathbf{h})$ satisfies the KKT conditions and it is optimal.

In order to better visualize how the generalized waterfilling is performed, we consider a simple example with $K = 2$ and with discrete joint channel states $\mathbf{h}^i$, $i = 1, \cdots, M$. Without loss of generality, let us assume $b_k(\mathbf{h}^1) < \cdots < b_k(\mathbf{h}^M)$. Figure 2 shows the generalized waterfilling procedure. The ordered values $b_k(\mathbf{h}^i)$ are illustrated in Figure 2(a). First, using (9), we solve for the amount of power $q_k(\mathbf{h}^1)$ that will level $L(\mathbf{h}^1, q_k(\mathbf{h}^1))$ and $b_k(\mathbf{h}^2)$, so that the water level is $b_k(\mathbf{h}^2)$, as shown in Figure 2(b). It can be easily shown that $q_k(\mathbf{h}^1)$ is the only non-negative solution to a k^{th} order polynomial equation, obtained from (9). In this particular example, the available average power is not yet completely used in this first step, so we repeat the same procedure at both $\mathbf{h}^1$ and $\mathbf{h}^2$, i.e., we solve for $q_k(\mathbf{h}^1)$ and $q_k(\mathbf{h}^2)$ that will level $L(\mathbf{h}^1, q_k(\mathbf{h}^1))$, $L(\mathbf{h}^2, q_k(\mathbf{h}^2))$ and $b_k(\mathbf{h}^3)$ (see Figure 2(c)). We continue this procedure until we see that although the water levels can be made equal at $b_k(\mathbf{h}^{t-1})$ while satisfying the average power constraint, it is not possible to equalize the water levels at $b_k(\mathbf{h}^t)$, since the available average power falls short of the required average power that is needed for such equalization. At this point, we know that the final water level, i.e., the true value of $1/\tilde{\lambda}_k$ that will satisfy the KKT conditions together with $q_k(\mathbf{h}^i)$ obtained from (9) should lie between $b_k(\mathbf{h}^{t-1})$ and $b_k(\mathbf{h}^t)$, and we can find it by searching between these two values until the $q_k(\mathbf{h}^i)$, $i = 1, \cdots, t-1$, satisfy the average power constraint with equality. Figure 2(d) illustrates this last step, and the final value of $\tilde{\lambda}_k$ that satisfies the KKT conditions.

Note that, by letting $\mu_1 = \cdots = \mu_K$, we recover the traditional waterfilling solution in [7, 8], since only the first term survives in the KKT conditions. On the other hand, if we let $\mathbf{s}_i = 1$ for $i = 1, \cdots, K$, the generalized waterfilling algorithm solves the resource allocation problem in [4] for scalar MAC.

4 Power Control with Limited Feedback

In Section 3, we have characterized the capacity region of the fading CDMA channel with the assumption of perfect channel state information at the transmitters and the receiver. This scenario is well suited for the theoretical treatment of the CDMA system, and in fact gives the utmost limit one can achieve in terms of reliable communication rates. On the other hand, perfect knowledge of the channel state at the transmitter is not a practical assumption, since it would require an infinite rate feedback link. The question that arises naturally is, should the side information be perfectly accurate in order for power control to be an effective means of increasing the capacity? Here, we will consider the power control problem from a more practical point of view, and demonstrate the effects of limited feedback on the set of achievable rates.

In particular, we will now consider the fact that the feedback link from the receiver to the transmitters is limited in rate, and therefore only part of the information that is available to the receiver can be communicated back to the transmitters to aid the resource allocation. On the other hand, we still assume that the feedback is instantaneous and error free.

Let us assume that the feedback link has the limitation that it allows reliable transmission of at most L bits per user. Then the receiver can inform the transmitter which one of up to 2^L transmit power levels to use, depending on the observed channel state. This requires a mapping from the channel state space to a discrete set of power levels, i.e., the problem of maximizing the weighted sum of rates as a function of a finite number of transmit power levels, can be formulated as a vector quantization problem

$$\begin{aligned} &\max \sum_{j=1}^{2^L} \int_{\gamma_j(\mathbf{h})} \sum_{i=1}^{K} (\mu_i - \mu_{i-1}) \log \left| \mathbf{I}_N + \sigma^{-2} \sum_{k=i}^{K} p_k(j) h_k \mathbf{s}_k \mathbf{s}_k^\top \right| f(\mathbf{h}) d\mathbf{h} \\ &\text{s.t.} \sum_{j=1}^{2^L} \int_{\gamma_j(\mathbf{h})} p_k(j) f(\mathbf{h}) d\mathbf{h} = \bar{p}_k \end{aligned} \tag{11}$$

where the channel state space, say $\mathbb{R}_+^K$, is partitioned into 2^L subsets $\gamma_j(\mathbf{h})$, and each of these partitions are mapped onto the element $p_k(j)$ from the codebook $\left\{p_k(1), \cdots, p_k(2^L)\right\}$. It is often a very tedious, if not impossible, task to find an optimal vector quantizer analytically for a given probability distribution of the quantizer input, even in the case of the traditional quantization where the goal is to represent a random vector as closely as possible. In fact, even much easier scalar quantization problems do not lend themselves to such solutions.

On the other hand, conditions for optimality of a quantizer lead to algorithmic solutions that yield "good" quantizers, which achieve local optima to the minimization problem [12]. Probably the most popular such algorithm is the Lloyd method which iterates between the partitioning and codebook selection [12].

It is possible to use the generalized Lloyd algorithm for vector quantization [12] to solve (11), by suitably defining an unconventional "distortion" function as the negative of the Lagrangian of the optimization problem in (11). However, this approach is still not guaranteed to obtain an absolutely optimal quantizer. In what follows, we will simplify the problem by limiting ourselves to scalar quantizers, where the goal is to represent the channel state, or optimal power level of each user as closely as possible. Namely, we will consider two settings: (i) a quantized version of the channel state information is fed back to the transmitters and the optimal power allocation is determined at the transmitters, and (ii) the optimal power levels are computed at the receiver, and are then quantized and fed back to the transmitters.

For the first case, let the quantizer $Q_i(h_i)$ be defined by the codebook $\hat{H}_i = \{\hat{h}_i^1 < \cdots < \hat{h}_i^{2^L}\}$, and the partition $W_i = \{0 = w_i^0 < w_i^1 < \cdots < w_i^{2^L-1} < \cdots < w_i^{2^L} = \infty\}$, such that

$$Q_i(h_i) = \hat{h}_i^j, \quad w_i^{j-1} \leq h_i < w_i^j, \quad j = 1, \cdots, 2^L, \quad i = 1, \cdots, K. \tag{12}$$

The quantization of a random variable is often performed subject to a fidelity criterion. In this particular case we consider the widely used mean square distortion as the fidelity criterion,

$$D(Q(h_i)) = E\left[(h_i - \hat{h}_i)^2\right], \quad i = 1, \cdots, K. \tag{13}$$

and we use Lloyd's algorithm [12,13] to perform the quantization of the variables of interest. Note that, although more advanced quantization techniques, including vector quantization, could be used to more accurately represent the original random variables, our purpose is to demonstrate how the power control performs for systems with quantized feedback when compared to ones with perfect (infinite precision) feedback, rather than finding a powerful quantizer. For our purposes, we will see that Lloyd's algorithm with the mean square distortion gives satisfactory enough results in terms of getting close to the perfect feedback capacity.

To avoid extra notation, we will assume that $Q_i(h_i)$ defined in (12) is a good quantizer obtained by running Lloyd's algorithm. When the channel state $\mathbf{h}$ is measured at the receiver, its components are quantized using $Q_i(h_i)$, $i = 1, \cdots, K$, and fed back to the transmitters. Then, for given priorities μ_i, the transmitters solve the optimal power allocation problem for the set of discrete channel states $\hat{h}_i^j$ using the generalized waterfilling algorithm, to get $p_i^*(\hat{\mathbf{h}})$, which yields a corresponding rate tuple $\hat{\mathbf{R}}$ given by the expectations in (2). One should note that the expectation in (2) is still with respect to the actual (unquantized) channel state $\mathbf{h}$.

For the second case, the quantizer for the power levels is similarly defined with the codebook $\hat{P}_i = \{\hat{p}_i^1 < \cdots < \hat{p}_i^{2^L}\}$, and the partition $v_i = \{0 = v_i^0 < v_i^1 < \cdots < v_i^{2^L-1} < \cdots < v_i^{2^L} = \infty\}$, such that

$$Q_i(p_i(\mathbf{h})) = \hat{p}_i^j, \quad v_i^{j-1} \leq p_i(\mathbf{h}) < v_i^j, \quad j = 1, \cdots, 2^L, \quad i = 1, \cdots, K. \tag{14}$$

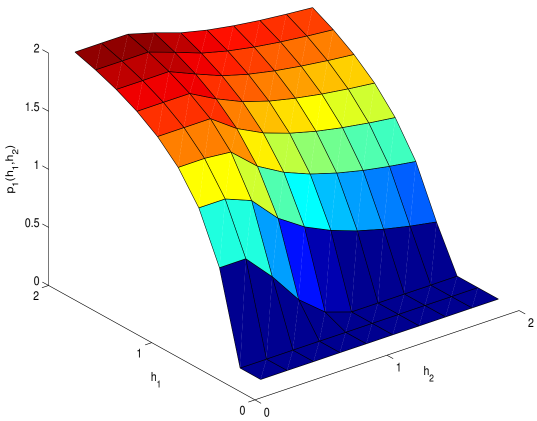
The mean square distortion function is also defined accordingly. Now, the receiver first uses the generalized waterfilling algorithm to obtain the optimal power levels $p_i^*(\mathbf{h})$ for given priorities, then quantizes them using $Q_i(p_i^*(\mathbf{h}))$, which is the quantizer obtained by the Lloyd's algorithm, and sends the quantized power level $\hat{p}_i^*$, to transmitter i, for $i = 1, \cdots, K$. These power levels can be used by the transmitters without the knowledge of the channel state, to obtain the rate tuple $\hat{\hat{\mathbf{R}}}$ which can be computed again by taking the expectation in (2).

The two feedback approaches differ in that while the total amount of receiver feedback is the same, the amount of feedback bits required by the transmitters is clearly different, since the transmitters need all KL feedback bits in case (i), whereas they need only their own power level, i.e., L bits, in case (ii). Also, the rate tuples $\hat{\mathbf{R}}$ and $\hat{\hat{\mathbf{R}}}$ are in general different. The corresponding achievable rate regions are given in the following section.

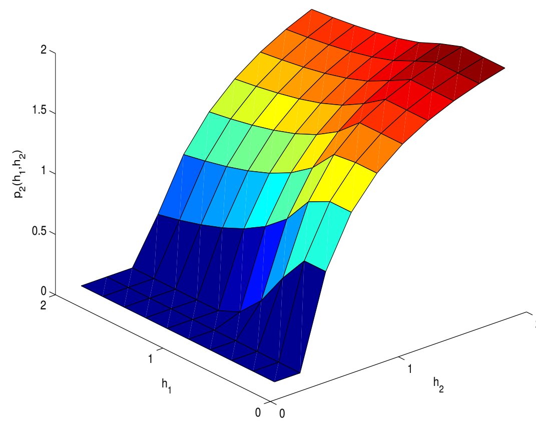
5 Simulation Results

In this section, we present some simulation results for the generalized iterative waterfilling algorithm. In our simulations, we pick the number of users $K = 2$, so that our results such as the capacity regions and the optimum power allocations can be easily visualized. The processing gain is chosen to be $N = 2$, the noise variance is $\sigma^2 = 1$, and both users have an average power constraint equal to 1.

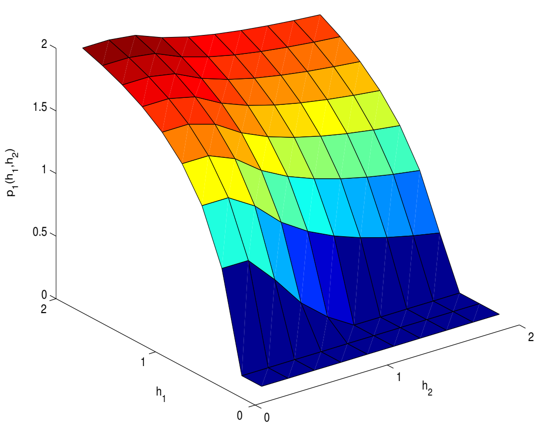
First, in order to observe the effect of the priorities μ_i on the optimum power allocation, we plot the optimum power allocation policies for both users for two different sets of (μ_1, μ_2) values. We fix the signature sequences of the users to be $\mathbf{s}_1 = [1/\sqrt{2}\ \ 1/\sqrt{2}]^\top$, and $\mathbf{s}_2 = [1\ \ 0]^\top$. The channel states h_1 and h_2 are chosen to be independent uniform random variables, each taking values from the discrete set $\{0.2{:}0.2{:}2\}$. Figures 3(a) and 3(b) correspond to the sum capacity maximizing power control policies, i.e., to $(\mu_1, \mu_2) = (1, 1)$. In each figure, the height of the surface corresponds to the power allocated to each channel state. We see that the two users perform simultaneous waterfilling, which was also observed in [7, 8]. Here, each of the users tend to transmit with less power over the channel states where the other user is stronger, and due to the symmetry of the problem, the power allocation policies are symmetric. When we choose unequal priorities $(\mu_1, \mu_2) = (1, 2)$, we observe in Figures 3(c) and 3(d) that the power allocation for user 1 does not change significantly, but



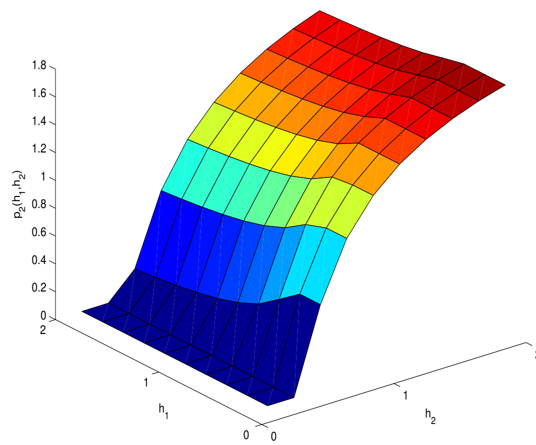
(a) Power allocation for user 1, $\mu_1 = \mu 2 = 1$.



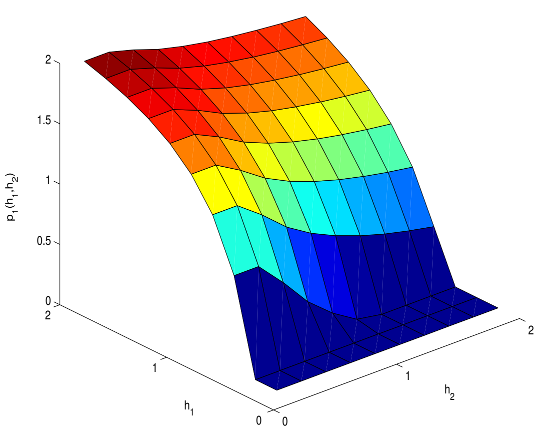
(b) Power allocation for user 2, $\mu_1 = \mu 2 = 1$.



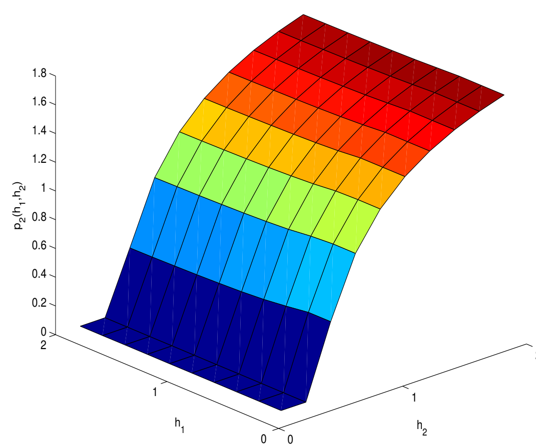
(c) Power allocation for user 1, $\mu_1 = 1$, $\mu 2 = 2$.



(d) Power allocation for user 2, $\mu_1 = 1$, $\mu 2 = 2$.



(e) Power allocation for user 1, $\mu_1 = 1$, $\mu 2 = 10$.



(f) Power allocation for user 2, $\mu_1 = 1$, $\mu 2 = 10$.

Figure 3: Power distributions for different values of priorities.

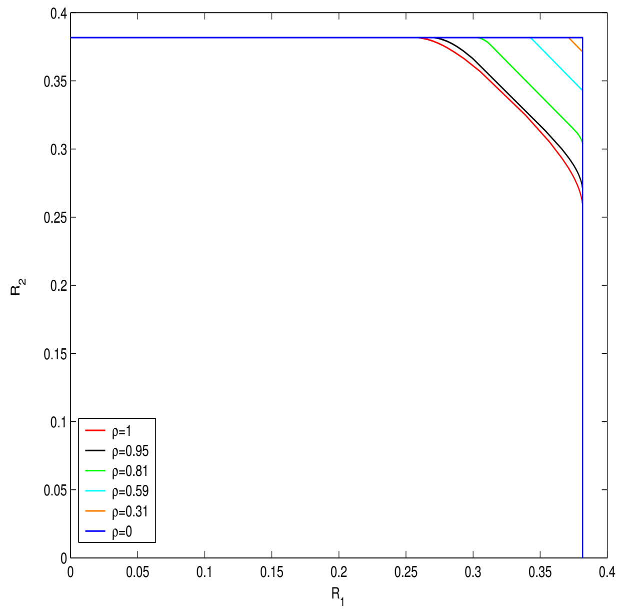


Figure 4: Capacity region of a two user fading CDMA channel for several correlation values among the signature sequences.

user 2 pours more power to channel states where it transmitted with considerably less power in the symmetric priorities case. If we increase μ_2 even further, and solve for the case when $(\mu_1, \mu_2) = (1, 10)$, we see in Figure 3(f) that the power allocation policy of user 2 converges nearly to single user waterfilling. Since the priority of user 1 is much less than that of user 2, user 2, while trying to maximize the weighted sum of rates, acts as if it is alone in the system in allocating its power. The power allocation of user 1, given in Figure 3(e), is not significantly different from the previous two cases.

In Figure 4, we give the capacity region of the fading CDMA channel, for different values of correlations between the signature sequences. The regions are formed by finding the optimal power allocation policies for a large set of (μ_1, μ_2) values, and then by using these allocation policies to compute the corresponding (R_1, R_2) pairs. The case when the correlation is $\rho = 1$ corresponds to the identical signature sequences case, in which case the boundary of the capacity region is strictly convex, and each point on the surface can be achieved by a power control policy, without timesharing. Note that, this setting also

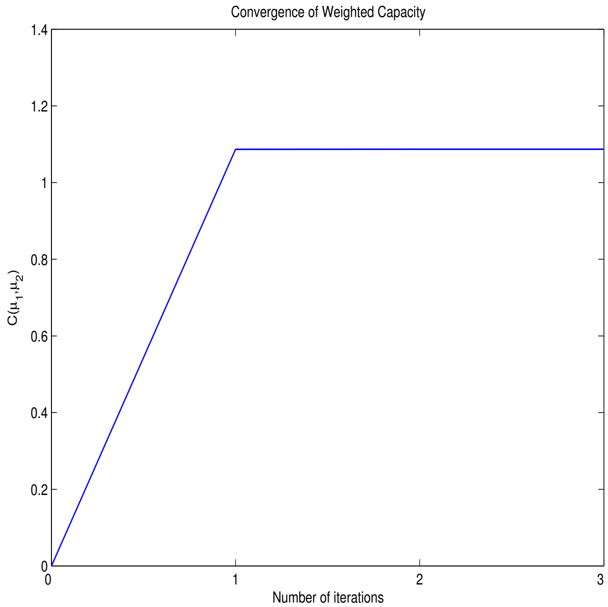


Figure 5: Convergence of the generalized iterative waterfilling algorithm.

covers the scalar MAC case in [4], and the properties derived in [4] for the capacity region are observed here. When we decrease the correlation between the sequences, we begin to observe a flat portion on the capacity region, which agrees with the findings of [7]. As we further decrease the correlation, eventually the sequences become orthogonal and the capacity region becomes the rectangular region whose boundaries are single user limits, as expected.

In Figure 5, we show an example of the convergence of the generalized iterative waterfilling algorithm for the simple system considered here; the powers converge after only three iterations, and the optimum weighted capacity value is almost attained after one round of iterations.

We now turn our attention to the systems in Section 4, and investigate the achievable rates for systems with limited feedback. Here, we consider i.i.d. Rayleigh channel fading (exponential channel states h_i) with mean 0.63. The signature sequences are fixed with correlation equal to $\rho = 0.95$.

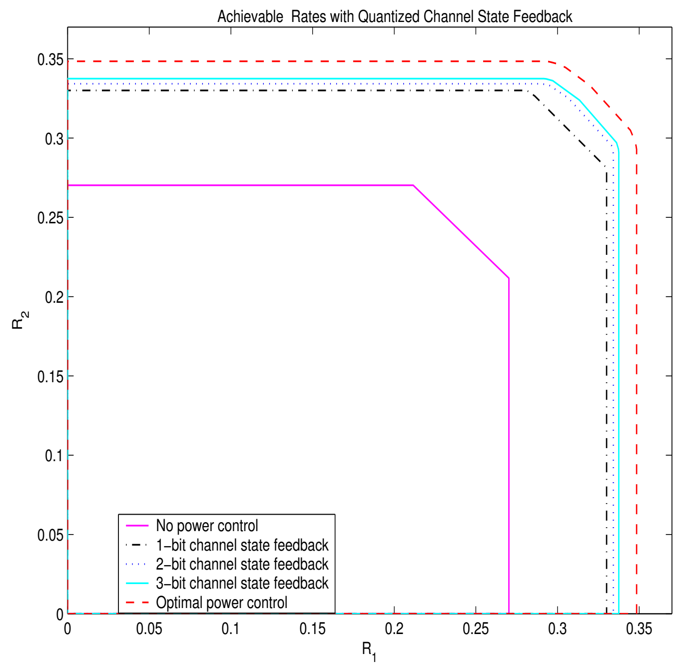


Figure 6: Achievable rates with quantized channel state feedback.

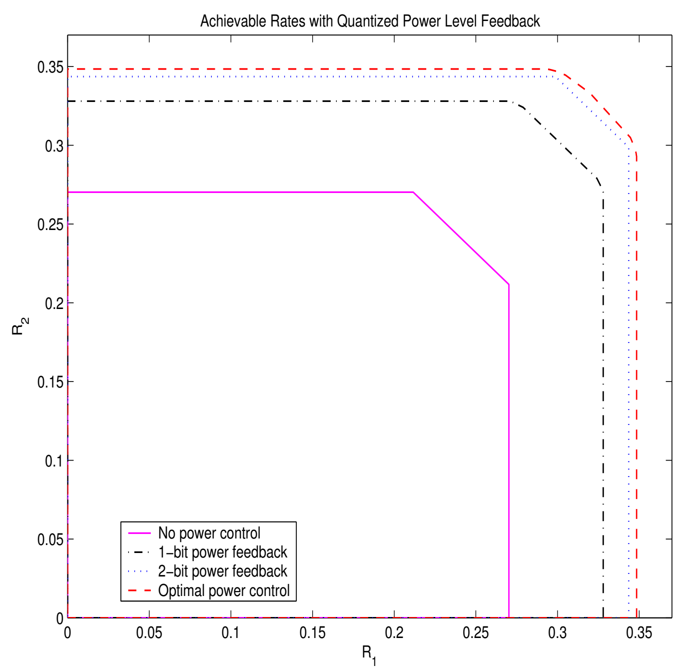


Figure 7: Achievable rates with quantized power level feedback.

The achievable rates for systems with 1, 2 and 3-bit quantization of each h_i are illustrated in Figure 6. We observe that, even with the very low feedback rate of 1-bit, the achievable rate region is dramatically improved when compared to a system with no feedback, thanks to the possibility of employing power control. We further see that, the amount of feedback, though it enlarges the region of achievable rates, is not very significant in terms of improving the set of achievable rates. We conclude that the feedback cost of power control may be kept to a minimum 1-bit per user, i.e., a total of K bits, while still achieving much higher information rates than no power control, in fact, rates very close to the perfect feedback capacity region. As discussed in Section 4, the set of achievable rates may further be improved by using more advanced quantization techniques.

For the case of directly feeding back a quantized power level, the achievable rates are shown in Figure 7. We observe that while 1-bit feedback gives similar achievable rates to those for channel state feedback, by 2-bit feedback we obtain a much more significant improvement when compared to the channel state feedback. In fact, by employing only four power levels, i.e., 2-bit feedback, it is possible to get very close to the capacity region with perfect feedback. Both feedback schemes show that the significant performance gains due to power control do not require very accurate feedback information, which is very promising in terms of possible implementation of the developed algorithm in practical systems.

6 Conclusions

We have characterized the power allocation policies that achieve arbitrary rate tuples on the boundary of the capacity region of a fading CDMA channel. The optimal power allocation policy for a given set of priorities μ_i is the joint solution to the extended KKT conditions for all users. Since the KKT inequalities appear difficult to solve analytically, we have provided a one-user-at-a-time iterative power allocation algorithm that converges to the optimum solution. We showed that, each iteration of this algorithm corresponds to solving for the power levels of the user of interest at all fading states, so that the power allocation satisfies the single-user KKT conditions. We have also provided a "generalized" waterfilling interpretation for the power allocation procedure as it operates by gradually equating the levels of "water" poured on top of certain base levels, which are functions of the channel states, power levels of other users, and the priorities μ_i. We then investigated the effect of limited feedback on the capacity region of the CDMA channel. We demonstrated that, even with very low rate feedback, rates very close to the boundary of the capacity region are achievable.

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