A reduced-order observer for the synchronization of Lorenz systems

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Abstract

This Letter presents a novel reduced-order nonlinear observer that synchronizes with a given Lorenz chaotic system. We prove the global exponential stability of the resulting error system and illustrate the result with a simulation.

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1. Introduction

Synchronization of chaotic systems has been an increasingly active area of research during the last decade [1,2]. Of particular interest is the connection between the observers for nonlinear dynamical systems and chaos synchronization by unidirectional coupling which is also known as master-slave configuration [3,4]. In this setup, a coupling signal from the master chaotic system is used to drive a slave system which is designed so as to yield estimates for the states of the master system. Thus, chaos synchronization problem can be posed as one of observer design where the coupling signal is viewed as the output and the slave system as the (reduced-order) observer. This view has enabled the application of many well-known results from nonlinear observer theory to chaos synchronization [5].

In this Letter we present a novel reduced-order observer structure to synchronize with a Lorenz system. The output of the master system is nonlinearly injected into the reduced-order observer dynamics in such a way that the resulting error system is globally exponentially stable.

The organization of the Letter is as follows. In Section 2 the reduced-order observer is presented and its convergence is proved; Section 3 illustrates the result with a simulation which is followed by concluding remarks.

2. Synchronization of Lorenz systems

Consider the Lorenz chaotic system described by the following set of differential equations:
\[ \begin{align*}
\dot{x} &= \sigma(y-x), \\
\dot{y} &= \rho x - y - xz, \\
\dot{z} &= xy - \beta z,
\end{align*} \tag{1}\]

where the system parameters \( \sigma, \rho, \beta > 0 \) are chosen so that the system (1) operates in chaotic regime. In the classical synchronization scheme we assume that the system (1) runs at the transmitter end and the state \( x \) is sent to the receiver via a communication channel as the synchronization signal \[ \text{[1]}. \]

Furthermore, it is assumed that the receiver has exact knowledge of the system parameters \( \sigma, \rho, \beta \).

Receiver’s task is to construct a dynamical system available signal \( x \) and the known parameters. To this end we propose the following nonlinear reduced-order observer:

\[ \begin{align*}
\dot{w} &= -\beta w + \left(1 - \frac{\beta}{2\sigma}\right)x^2, \\
\dot{\hat{y}} &= -\dot{\hat{y}} + \rho x - x\hat{z}, \\
\hat{z} &= w + \frac{x^2}{2\sigma},
\end{align*} \tag{2a-c}\]

where the signals \( w \) and \( \dot{\hat{y}} \) can be viewed as the states of the observer. Next we show the convergence of the observer states to those of the transmitter chaotic system.

**Theorem 1.** When the parameters \( \sigma, \rho, \beta \) are chosen so that (1) operates in chaotic regime, the receiver system (2) yields exponentially converging estimates for the signals \( y \) and \( z \).

**Proof.** Since the system (1) is chaotic, all of its signals remain bounded. In particular, there exists a positive number \( A \) such that \( |x(t)| < A, \forall t \geq 0 \). Let us choose a Lyapunov function candidate

\[ V = k_1 \varepsilon_y^2 + k_2 \varepsilon_z^2, \]

where \( \varepsilon_y = \hat{y} - y, \varepsilon_z = \hat{z} - z \) and \( k_1, k_2 > 0 \) are chosen to satisfy

\[ k_2 = \frac{A^2 k_1 + \epsilon/k_1}{4\beta}, \quad \epsilon > 0. \tag{3} \]

The derivative of \( V \) along the trajectories of (1) and (2) satisfies

\[ \begin{align*}
\dot{V} &= 2k_1 \varepsilon_y \varepsilon_y + 2k_2 \varepsilon_z \hat{z} \\
&= 2k_1 \varepsilon_y (-\dot{\hat{y}} + \rho x - x\hat{z} - \rho x + y + xz) \\
&\quad + 2k_2 \varepsilon_z \left( \dot{w} + \frac{2x\dot{\hat{y}}}{2\sigma} - \hat{z} \right) \\
&= -2k_1 \varepsilon_y^2 - 2k_2 \varepsilon_z^2 - 2k_1 x \varepsilon_y \varepsilon_z \\
&\leq -2k_1 \varepsilon_y^2 - 2k_2 \varepsilon_z^2 - 2k_1 A \varepsilon_y \varepsilon_z \\
&= -2k_1 \varepsilon_y^2 - 2k_2 \varepsilon_z^2 - 2k_1 A \varepsilon_y \varepsilon_z \\
&= -[\varepsilon_y \varepsilon_z] Q[\varepsilon_y \varepsilon_z]^T,
\end{align*} \]

where

\[ Q = \begin{bmatrix} 2k_1 & k_1 A \\ k_1 A & 2k_2 \beta \end{bmatrix}. \]

By using (3), it can be shown that

\[ \text{det}(Q) = 4\beta k_1 k_2 - k_1^2 A^2 = 4k_1 \beta \left( \frac{A^2 k_1 + \epsilon/k_1}{4\beta} \right) - k_1^2 A^2 = A^2 k_1^2 + \epsilon - k_1^2 A^2 = \epsilon. \]

Therefore, the matrix \( Q \) is positive definite which implies that there exists positive constants \( M, \alpha \) such that the vector of synchronization error \( \varepsilon = [\varepsilon_y \varepsilon_z]^T \) satisfies

\[ \|\varepsilon(t)\| = \|\varepsilon(0)\| Me^{-\alpha t}, \quad t \geq 0. \]

Hence, the receiver system synchronizes with the transmitter system exponentially fast \[ \text{[6]} \].

**3. Simulation**

For the illustration of the proposed synchronization method we chose a Lorenz system with the parameters \( \sigma = 10, \rho = 28 \) and \( \beta = 8/3 \). The convergence of the estimates to the true signals is shown in Fig. 1.
4. Conclusion

In this Letter we have given a novel reduced order nonlinear observer to synchronize Lorenz chaotic systems. We have shown that the states of the observer exponentially converge to those of the original system.

References