IŞIK UNIVERSITY, MATH 230 MIDTERM EXAM II

Q1 |	**Q2** |	**Student ID:**	**Row No:**
Q3 |		**Bonus Q1** |	**Bonus Q2** |
Last Name:		**First Name:**	

I pledge my honour that I have not violated the honour code during this examination. Bu sınavda onur yasamızı ihlal etmediğime şerefim üzerine yemin ederim.	**Signature :**

1. (16 points) In a factory producing soda, the bottles are filled according to the Gaussian distribution with mean 300 ml and variance 100 ml. If each bottle has capacity of 315 ml, what percentage of the bottles are overfilled (filled above the capacity)?

Solution :

Define

$$X = \text{volume of soda in a bottle}$$

Then

$$X \sim N(300, 100).$$

For a bottle to be overfilled, $X > 315$. So

$$\mathbb{P}(X > 315) = \mathbb{P}\left(\frac{X-300}{10} > \frac{315-300}{10}\right) = \mathbb{P}(Z > 1.5)$$

$$= 1 - \mathbb{P}(Z \leq 1.5) = 1 - \Phi(1.5) = 1 - 0.9332.$$

2. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, and A_i denote events in a sample space S and let $\mathbb{P}(.)$ denote a probability measure on S.
(*Note: A statement is assumed to be true if it is true* **in any possible case**, *and it is assumed to be false if it fails in at least one case.*):

i. For a continuous random variable, probabilty mass function is always zero. T *F*

ii. PDF values must be less than 1. T *F*

iii. In a Bernouli trial there are 3 possible outcomes. *T* F

iv. Probabilities can be computed using the random variable's CDF. T *F*

v. Lifetime of computer part is a continuous random variable. T *F*

vi. For a standard Gaussian X, we have $\mathbb{P}(X > 1) > 0.5$. *T* F

vii. Probability mass function is always positive. T *F*

viii. For any random variable X, we have $\mathbb{P}(X \in A) = 1 - \mathbb{P}(X \in A^c)$. T *F*

ix. For any random variable X, we have $\mathbb{E}(X^2) \geq \mathbb{E}^2(X)$. T *F*

x. Number of sunny days in a week is a binomial random variable. T *F*

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3. (16 points) Let X be a continuous random variable with the probability density function (PDF)

$$f(x) = \begin{cases} ax + b & , 0 \le x \le 1 \\ 0 & , else \end{cases}$$

If it is given that the expectation of X is $7/12$, then

i. find the values of a and b.
Solution :

First,

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 (ax+b)dx = \frac{ax^2}{2} + bx\Big]_0^1 = \frac{a}{2} + b.$$

Second,

$$\frac{7}{12} = \mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 (ax^2 + bx)dx = \frac{a}{3} + \frac{b}{2}.$$

Solve the system of two equations

$$\frac{a}{2} + b = 1, \quad \frac{a}{3} + \frac{b}{2} = \frac{7}{12}.$$

Then $a = 1$ and $b = 1/2$.

ii. Evaluate the probability $\mathbb{P}(1/2 \le X < 2)$.
Solution :

Since

$$f(x) = \begin{cases} x + \frac{1}{2} & , 0 \le x \le 1, \\ \\ 0 & , \text{else}, \end{cases}$$

we have

$$\mathbb{P}(1/2 \le X < 2) = \int_{1/2}^{2} f(x)dx = \int_{1/2}^{1} \left(x + \frac{1}{2}\right) dx$$
$$= \frac{x^2}{2} + \frac{x}{2}\Big]_{1/2}^{1} = 1 - \frac{3}{8} = \frac{5}{8}.$$

BONUS - Q1 (15 points) Find the value of the integral

$$\int_{-1}^{5} e^{-(x-4)^2/32} dx$$

Solution :

We note that the probability of a normal random variable with parameters μ and σ^2 being between a and b is

$$\mathbb{P}(a < X < b) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-(x-\mu)^2/2\sigma^2} dx.$$

Hence define $X \sim N(4, 16)$. Then this integral equals

$$\begin{aligned}
&= \sqrt{32\pi} \cdot \mathbb{P}(-1 < X < 5) \\
&= \sqrt{32\pi} \cdot \mathbb{P}\left(\frac{-1-4}{4} < \frac{X-4}{4} < \frac{5-4}{4}\right) \\
&= \sqrt{32\pi}\,(\Phi(0.25) - \Phi(-1.25)) \\
&= \sqrt{32\pi}\,(\Phi(0.25) - 1 + \Phi(1.25)) \\
&= \sqrt{32\pi}(0.5987 + 0.8944 - 1).
\end{aligned}$$

BONUS - Q2 (15 points) Let X be a Uniform random variable on (a,4) for some $a < -1$. If

$$\mathbb{P}(X^2 - X - 2 < 0) = 1/2$$

then what is the value of a?

Solution :

If $X \sim Uniform(a, 4)$ then its PDF is

$$f(x) = \begin{cases} \dfrac{1}{4-a} & , a \leq x \leq 4, \\ 0 & , \text{else}. \end{cases}$$

Also

$$\begin{aligned}
\frac{1}{2} &= \mathbb{P}(X^2 - X - 2 < 0) = \mathbb{P}((X-2)(X+1) < 0) \\
&= \mathbb{P}(-1 < X < 2) \\
&= \int_{-1}^{2} \frac{1}{4-a} dx = \left.\frac{x}{4-a}\right]_{-1}^{2} = \frac{3}{4-a}.
\end{aligned}$$

Hence

$$4 - a = 6$$

and

$$a = -2.$$

Q4 |	**Q5** |	**Student ID:**	**Row No:**
Q6 |	**Q7** |		
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4. (14 points)There are 6 balls in an urn numbered 1 through 6. You randomly select 4 of those balls. Let the random variable Y denote the maximum of the four numbers on the selected balls. Find the sample space S_Y of Y and the probability mass function of Y.

Solution :
Note that $S_Y = \{4, 5, 6\}$. Then

$$p(4) = \mathbb{P}(Y = 4) = \frac{1}{\binom{6}{4}}, \quad p(5) = \mathbb{P}(Y = 5) = \frac{\binom{4}{3}}{\binom{6}{4}}, \quad p(6) = \mathbb{P}(Y = 6) = \frac{\binom{5}{3}}{\binom{6}{4}}.$$

$$p(x) = \begin{cases} \dfrac{1}{\binom{6}{4}} & , x = 4, \\[2ex] \dfrac{\binom{4}{3}}{\binom{6}{4}} & , x = 5, \\[2ex] \dfrac{\binom{5}{3}}{\binom{6}{4}} & , x = 6, \\[2ex] 0 & , \text{else.} \end{cases}$$

5. (15 points) You start with a full deck of cards. You select cards from the deck, with replacement, until you get **a card other than an ace**. (There are 52 cards in a standard deck, out of which 4 are Aces.)

   i. Find the probability that at most 3 cards should be drawn?

   Solution :
   Define

$$X = \text{the first trial that you obtain a card other than ace.}$$

   Then

$$X \sim Geometric\left(\frac{48}{52}\right).$$

   The question is

$$\mathbb{P}(X \leq 3) = p(1) + p(2) + p(3) = \frac{48}{52} + \frac{4}{52} \cdot \frac{48}{52} + \left(\frac{4}{52}\right)^2 \cdot \frac{48}{52}.$$

   ii. What is the expected value of the number of cards drawn?

   Solution :

$$\mathbb{E}(X) = \frac{1}{\frac{48}{52}} = \frac{52}{48}.$$

   iii. What is the variance of the number of cards drawn?

   Solution :

$$Var(X) = \frac{1}{\left(\frac{48}{52}\right)^2} - \frac{1}{\left(\frac{48}{52}\right)} = \frac{\frac{4}{52}}{\left(\frac{48}{52}\right)^2}.$$

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6. (15 points) Suppose that the **annual** number of storms that are formed off the Antalya coast has a Poisson distribution with mean 15. What is the probability that during **6 months (half a year)** there are **at most** five storms?

Solution :
Define

$$X = \text{ the number of storms at Antalya coast during } \underline{6 \text{ months}}.$$

Then X is a *Poisson* random variable with parameter $\lambda = \dfrac{15}{2}$. Hence

$$\mathbb{P}(X \leq 5) = p(0) + p(1) + p(2) + p(3) + p(4) + p(5)$$

$$= e^{-15/2} + e^{-15/2} \cdot \frac{15}{2} + e^{-15/2} \cdot \frac{\left(\frac{15}{2}\right)^2}{2} + e^{-15/2} \frac{\left(\frac{15}{2}\right)^3}{3!}$$

$$+ e^{-15/2} \cdot \frac{\left(\frac{15}{2}\right)^4}{4!} + e^{-15/2} \cdot \frac{\left(\frac{15}{2}\right)^5}{5!}.$$

7. (14 points) We play a game in the following way: we roll four dice one after the other and if a die lands on six we win 5 dollars and if it lands on three we lose 4 dollars (and nothing happens for the other numbers). The four dice are biased in the following way :

Die 1: $p(6) = 1/4$, $p(5) = 1/4$,
and all other numbers are equally probable.

Die 2: $p(6) = 2/9$, $p(5) = 2/9$,
and all other numbers are equally probable.

Die 3: $p(6) = 1/5$, $p(5) = 1/5$,
and all other numbers are equally probable.

Die 4: $p(6) = 2/11$, $p(5) = 2/11$,
and all other numbers are equally probable.

What is the expected amount of money in our pocket at the end of the game. Solve the problem considering the expected value of sum of random variables.

Solution :
Define

$$\begin{aligned} X_1 &= \text{ winnings from die 1,} \\ X_2 &= \text{ winnings from die 2,} \\ X_3 &= \text{ winnings from die 3,} \\ X_4 &= \text{ winnings from die 4.} \end{aligned}$$

Then

$$\mathbb{E}(X_1) = 5 \cdot p(6) - 4 \cdot p(3) = 5 \cdot \frac{1}{4} - 4 \cdot \frac{1}{8} = \frac{3}{4}$$

$$\mathbb{E}(X_2) = 5 \cdot p(6) - 4 \cdot p(3) = 5 \cdot \frac{2}{9} - 4 \cdot \frac{5}{36} = \frac{5}{9}$$

$$\mathbb{E}(X_3) = 5 \cdot p(6) - 4 \cdot p(3) = 5 \cdot \frac{1}{5} - 4 \cdot \frac{3}{20} = \frac{2}{5}$$

$$\mathbb{E}(X_4) = 5 \cdot p(6) - 4 \cdot p(3) = 5 \cdot \frac{2}{11} - 4 \cdot \frac{7}{44} = \frac{3}{11}$$

And

$$\begin{aligned}\mathbb{E}(X_1 + X_2 + X_3 + X_4) &= \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) \\ &= \frac{3}{4} + \frac{5}{9} + \frac{2}{5} + \frac{3}{11}.\end{aligned}$$