Last Updated: 22.02.2017 May 12, 2016

IŞIK UNIVERSITY, MATH 230 FINAL EXAM

Q1 |	**Q2** |	**Student ID:**	**Row No:**
Q3 |	**Q4** |		
Last Name:		**First Name:**	

I pledge my honour that I have not violated the honour code during this examination. Bu sınavda onur yasamızı ihlal etmediğime şerefim üzerine yemin ederim.	**Signature :**

1. (15 points) 70% of the light aircraft that disappear while in flight are discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft **not** discovered do **not have** any emergency locator. Suppose that a light aircraft has disappeared. If it has an emergency locator, what is the probability that it will be discovered?

   Solution :

   Define the events

   $$D = \text{ airplane discovered,}$$
   $$E = \text{ airplane has an emergency locator.}$$

   Then we are given that

   $$\mathbb{P}(D) = 0.7,$$
   $$\mathbb{P}(E|D) = 0.6,$$
   $$\mathbb{P}(E^c|D^c) = 0.9.$$

   The question is

   $$\mathbb{P}(D|E) = \frac{\mathbb{P}(E|D)\cdot\mathbb{P}(D)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|D)\cdot\mathbb{P}(D)}{\mathbb{P}(E|D)\cdot\mathbb{P}(D)+\mathbb{P}(E|D^c)\cdot\mathbb{P}(D^c)}$$
   $$= \frac{(0.6)\cdot(0.7)}{(0.6)\cdot(0.7)+(1-0.9)\cdot(1-0.7)}$$

   by Bayes' theorem.

2. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, and A_i denote events in a sample space S and let $\mathbb{P}(.)$ denote a probability measure on S.
(*Note: A statement is assumed to be true if it is true* **in any possible case**, *and it is assumed to be false if it fails in at least one case.*):

i. If the joint density function of two dependent random variables is given, then the marginal densities can be calculated. (T) *F*

ii. If two random variables X and Y are independent, then for the densities we have $f_X(x)f_Y(y) = f_{X,Y}(x,y)$. (T) *F*

iii. Joint CDF of two random variables can be negative. *T* (F)

iv. If X and Y are 2 jointly continuous random variables then their joint PDF satisfies $0 \leq f_{X,Y}(x,y) \leq 1$. *T* (F)

v. If X and Y are 2 discrete random variables then their joint PMF satisfies $0 \leq p_{X,Y}(x,y) \leq 1$. (T) *F*

vi. If X and Y are 2 discrete random variables then their joint PMF $p_{X,Y}$ is the product of their marginal PMFs. *T* (F)

vii. For any two jointly contiuous X and Y, we have $f_{X|Y} = f_X$. *T* (F)

viii. For any two jointly contiuous X and Y, we have $\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = f_X(x)$. (T) *F*

ix. If the joint CDF of X and Y is given by $F_{X,Y} = (1-e^{-x})(1-e^{-y})$ for $x > 0, y > 0$, then the marginal PDF of X is $f_X(x) = e^{-x}$ for $x > 0$. (T) *F*

x. If X and Y are 2 discrete random variables then $\sum_x \sum_y p_{X,Y}(x,y) = 1$. (T) *F*

Student's Name :________________

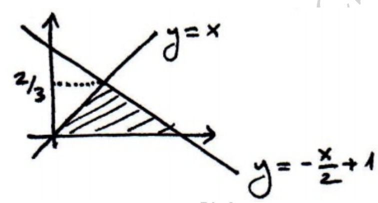
3. (15 points) Suppose

$$f_{X,Y}(x,y) = \begin{cases} 3xy & ,0 < y < -\frac{x}{2}+1, \quad x > 0, \\ 0 & \text{else} \end{cases}$$

is the joint density function of the random variables X and Y.

i. Find the probability $\mathbb{P}(X > Y)$.

Solution :



$$\begin{aligned}\mathbb{P}(X > Y) &= \int_{y=0}^{y=2/3}\int_{x=y}^{x=2-2y} 3xy\,dxdy = \int_{y=0}^{y=2/3} \frac{3}{2}x^2y\Big]_{x=y}^{x=2-2y} dy \\ &= \int_{y=0}^{y=2/3} \frac{3}{2}y[(2-2y)^2 - y^2]dy = \int_{y=0}^{y=2/3} \frac{3}{2}(4y - 8y^2 + 3y^3)dy \\ &= \frac{3}{2}\left(2y^2 - \frac{8y^3}{3} + \frac{3y^4}{4}\right)\Big]_{y=0}^{y=2/3} = \frac{10}{27}.\end{aligned}$$

ii. Find the marginal density, $f_X(x)$, of X.

Solution :

$$\begin{aligned}f_X(x) &= \int_{y=0}^{y=-\frac{x}{2}+1} 3xy\,dy = \frac{3}{2}xy^2\Big]_{y=0}^{y=-\frac{x}{2}+1} \\ &= \frac{3}{2}x(-\frac{x}{2}+1)^2 = \frac{3}{8}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x, \quad 0 < x < 2.\end{aligned}$$

iii. Find the conditional density $f_{Y|X}(y|1)$.

Solution :

$$f_{Y|X}(y|1) = \frac{f_{Y,X}(y,1)}{f_X(1)} = \frac{3y}{3/8} = 8y, \quad 0 < y < 1.$$

4. (12 points)

i. The height of pine trees of age 10 are estimated to be a normal distribution with the mean 200 cm and variance 64 cm. What is the probability a pine tree of age 10 grows more than 210 cm?

Solution :
Let

$$X = \text{ height of pine tree of age 10.}$$

Then

$$X \sim N(200, 64).$$

So

$$\mathbb{P}(X > 210) = \mathbb{P}\left(\frac{X-200}{8} > \frac{210-200}{8}\right)$$

$$= \mathbb{P}(Z > 1.25) = 1 - \mathbb{P}(Z \leq 1.25)$$

$$= 1 - \phi(1.25) = 1 - 0.8944$$

ii. Suppose that the amount of time a repairman needs to fix a furnace is uniformly distributed between 1.5 and 4 hours. What is the probability that fixing a certain furnace will last less than 2 hours?

Solution :
Let

$$X = \text{amount of time (in hours) needed to fix a furnace.}$$

Then

$$X \sim Uniform(1.5, 4).$$

So its PDF is

$$f(x) = \begin{cases} \frac{1}{2.5} & , 1.5 < x < 4, \\ 0 & , else. \end{cases}$$

Hence

$$\mathbb{P}(X < 2) = \int_{1.5}^{2} \frac{1}{2.5} dx = \frac{0.5}{2.5} = \frac{1}{5}.$$

Q5 |	Q6 |	Student ID:	Row No:
Q7 |	Q8 |	Bonus |	
Last Name:		First Name:	

5. (12 points)

   i. A random sample of 45 students from different universities were selected randomly and asked whether they are happy with their social activities on campus. The responses of 32 were negative. If Ahmet, Ali and Zeynep were among those questioned, what is the probability that all three of them gave negative responses?

   Solution :

$$\frac{\binom{32}{3}}{\binom{45}{3}}$$

   ii. In a class, 11 of the 25 students dislike rock music, eight like rock music, and the rest are indifferent. A random sample of five students is selected for an interview. What is the probability that all of them have the same opinion about rock music ?

   Solution :
   We need to consider cases where all like rock music, all dislike rock music and all are indifferent. So

$$\frac{\binom{11}{5}}{\binom{25}{5}} + \frac{\binom{8}{5}}{\binom{25}{5}} + \frac{\binom{6}{5}}{\binom{25}{5}}$$

6. (15 points) Let X be a continuous random variable with probability density

$$f(x) = c\,e^{-3x} \quad , x > 0$$

for some constant c.

i. Find the value of c.

Solution :

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} ce^{-3x}dx = -\frac{c}{3}e^{-3x}\Big]_0^{\infty} = \frac{c}{3}(1-0).$$

Hence $c = 3$.

ii. What is the probability $\mathbb{P}(1 < X < 3)$?

Solution :

$$\mathbb{P}(1 < X < 3) = \int_1^3 3e^{-3x}dx = -e^{-3x}\big]_1^3 = e^{-3} - e^{-9}.$$

iii. Find the expectation of X.

Solution :

$$\mathbb{E}(X) = \int_0^{\infty} x \cdot 3e^{-3x}dx$$

Let

$$u = x, \qquad dv = 3e^{-3x}dx,$$
$$du = dx, \qquad v = -e^{-3x}.$$

So

$$\mathbb{E}(X) = -xe^{-3x}\big]_0^{\infty} + \int_0^{\infty} e^{-3x}dx = 0 - \frac{e^{-3x}}{3}\Big]_0^{\infty} = \frac{1}{3}.$$

Student's Name :________________

7. (10 points) Urn I contains three "25 kuruş" and four "50 kuruş", urn II contains two "25 kuruş" and five "50 kuruş", and urn III contains three "25 kuruş" and one "50 kuruş". One coin is selected at random from each urn. If two of the three coins are "50 kuruş" coins, what is the probability that the coin selected from urn I is a "50 kuruş" coin ?

Solution :

Let's call

$$U_1 = \text{the coin selected from urn 1},$$
$$U_2 = \text{the coin selected from urn 2},$$
$$U_3 = \text{the coin selected from urn 3}.$$

Then
$\mathbb{P}(U_1 = 50 | 2 \text{ coins are 50 kr.})$

$$= \frac{\mathbb{P}(U_1 = 50 | 2 \text{ coins are 50 kr.})}{\mathbb{P}(2 \text{ coins are 50 kr.})}$$

$$= \frac{\mathbb{P}(U_1 = 50, U_2 = 50, U_3 = 25) + \mathbb{P}(U_1 = 50, U_2 = 25, U_3 = 50)}{\mathbb{P}(U_1 = 50, U_2 = 50, U_3 = 25) + \mathbb{P}(U_1 = 50, U_2 = 25, U_3 = 50) + \mathbb{P}(U_1 = 25, U_2 = 50, U_3 = 50)}$$

$$= \frac{\frac{4}{7} \cdot \frac{5}{7} \cdot \frac{3}{4} + \frac{4}{7} \cdot \frac{2}{7} \cdot \frac{1}{4}}{\frac{4}{7} \cdot \frac{5}{7} \cdot \frac{3}{4} + \frac{4}{7} \cdot \frac{2}{7} \cdot \frac{1}{4} + \frac{3}{7} \cdot \frac{5}{7} \cdot \frac{1}{4}}$$

$$= \frac{68}{83}.$$

8. (11 points)

i. Suppose you and your friend flip coins and play a game. Each of you flip a fair coin. If the upper faces match (they are the same), and they are tails then you win \$1. If the upper faces match and they are heads, you win \$2. You loose \$1.50 if the coins do not match (if one is head and the other is tail). What is your expected winnings ?

Solution :

Let

$$X = \text{your winnings in dollars.}$$

Then

$$S_X = \{1, 2, -1.5\}$$

and

$$p(x) = \begin{cases} \frac{1}{4} & , x = 1, \\ \frac{1}{4} & , x = 2, \\ \frac{1}{2} & , x = -1.5, \\ 0 & , else. \end{cases}$$

Then

$$\mathbb{E}(X) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} - (1.5) \cdot \frac{1}{2} = 0\,\$$$

BONUS (10 points) Julia is interested in two games, Beno and Kolita. To play Kolita, she buys a ticket for \$1, draws a ball at random from a box of 100 balls numbered 1 to 100. If the ball drawn matches the number on her ticket, she wins \$75; otherwise, she loses. To play Beno, Julia bets \$1 on a single number that has a 25% chance to win. If she wins, they will return her dollar plus two dollars more; otherwise, they keep the dollar. Calculate the expected value and variances of both games.

Solution :

Let

$$K = \text{ winnings in Kolita.}$$

Then its PMF is

$$p(x) = \begin{cases} \frac{99}{100} & , x = -1 \\ \frac{1}{100} & , x = 74 \\ 0 & , else \end{cases}$$

So

$$\mathbb{E}(K) = -1 \cdot \frac{99}{100} + 74 \cdot \frac{1}{100} = -\frac{25}{100}$$

$$\mathbb{E}(K^2) = 1 \cdot \frac{99}{100} + (74)^2 \cdot \frac{1}{100} = \frac{74^2 + 1}{100}$$

$$Var(K) = \frac{74^2 + 1}{100} - \left(-\frac{25}{100}\right)^2$$

Next, let

$$B = \text{ winnings in Beno.}$$

Then its PMF is

$$p(x) = \begin{cases} \dfrac{1}{4} & , x = 2 \\ \\ \dfrac{3}{4} & , x = -1 \\ \\ 0 & , else \end{cases}$$

So

$$\mathbb{E}(B) = 2 \cdot \frac{1}{4} - 1 \cdot \frac{3}{4} = -\frac{1}{4}$$

$$\mathbb{E}(B^2) = 4 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{7}{4}$$

$$\begin{aligned} Var(B) &= \frac{7}{4} - \left(-\frac{1}{4}\right)^2 \\ &= \frac{7}{4} - \frac{1}{16}. \end{aligned}$$