December 1, 2016

IŞIK UNIVERSITY, MATH 230 MIDTERM EXAM

Q1 |	Q2 |	Student ID:	Row No:
Q3 |			
Last Name:		First Name:	

1. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, and A_i denote events in a sample space S and let $\mathbb{P}(.)$ denote a probability measure on S.
(*Note: A statement is assumed to be true if it is true* **in any possible case**, *and it is assumed to be false if it fails in at least one case.*):

i.	For any random variable X, we have $\mathbb{E}(X^2) = (\mathbb{E}(X))^2$.	T	(F)
ii.	If $\mathbb{P}(A\|B) = \mathbb{P}(A)$ then $\mathbb{P}(B\|A) = \mathbb{P}(B)$.	(T)	F
iii.	$\mathbb{P}(A\|B^c) = 1 - \mathbb{P}(A\|B)$.	T	(F)
iv.	Always, $\mathbb{E}(X) \geq 0$.	T	(F)
v.	If $\mathbb{P}(A\|B) = \mathbb{P}(B\|A)$ then A and B are equally likely events	T	(F)
vi.	If $F(x)$ is a CDF, then $F(x) \geq 0$.	(T)	F
vii.	If $\mathbb{P}(A) = (\mathbb{P}(A))^2$ then A is independent of A.	(T)	F
viii.	The sum of all coefficients in the expansion of $(x+y)^{12}$ is 2^{12}.	(T)	F
ix.	The coefficient of x^2y^5z in the expansion of $(x+y+z)^8$ is 168.	(T)	F
x.	If $\mathbb{P}(A) \leq \mathbb{P}(B)$ then $A \subset B$.	T	(F)

2. (20 points) You play a game with a fair die. You roll the die once. If the die shows an even number, you win 3 TL, if the die shows 1 or 3 then you loose 4 TL and if the die shows a 5 you win 1 TL. Let X denote your winnings/losses after one turn of this game.

   i. What is the PMF (probability mass function) of X?
   Solution :
   Note that $S_X = \{3, -4, 1\}$ and the PMF is

$$p(x) = \begin{cases} \frac{1}{2} & , x = 3, \\ \frac{1}{6} & , x = 1, \\ \frac{1}{3} & , x = -4, \\ 0 & , \text{else.} \end{cases}$$

   ii. Find your expected winnings/losses after one turn of this game.
   Solution :

$$\begin{aligned} \mathbb{E}(X) &= 3 \cdot p(3) + 1 \cdot p(1) - 4 \cdot p(-4) \\ &= 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{6} + (-4) \cdot \frac{1}{3} \\ &= \frac{1}{3}. \end{aligned}$$

   iii. What is the variance of winnings/losses after one turn of this game.
   Solution :

$$\begin{aligned} \mathbb{E}(X^2) &= 3^2 \cdot p(3) + 1^2 \cdot p(1) + (-4)^2 \cdot p(-4) \\ &= 3^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{6} + (-4)^2 \cdot \frac{1}{3} \\ &= 10. \end{aligned}$$

$$Var(x) = 10 - \left(\frac{1}{3}\right)^2 = \frac{89}{9}.$$

Student's Name : ______________

3. (20 points) A family of 5 people will decide whether they paint their house red or green. Each member of the family will vote for one of these colors, and the color with majority's votes will be the new color of the house. Family members randomly chooses between red and green, and each chooses red with probability 0.7.

   i. What is the probability that this house will be painted green?

      Solution :
      Let us define the random variable

      $$X = \text{the number of votes for green.}$$

      Then we observe that

      $$X \sim \mathit{Binomial}(5, 0.3).$$

      Hence the probability that the majority of the votes are green is

      $$\begin{aligned}\mathbb{P}(X \geq 3) &= p(3) + p(4) + p(5) \\ &= \binom{5}{3}(0.3)^3(0.7)^2 + \binom{5}{4}(0.3)^4(0.7)^1 + \binom{5}{5}(0.3)^5(0.7)^0.\end{aligned}$$

   ii. What is the expected number of votes for green?
      Solution :
      Using the expectation of a binomial random varable,

      $$\mathbb{E}(X) = n \cdot p = 5(0.3) = 1.5 \text{ votes.}$$

Q4 |	Q5 |	Student ID:	Row No:
Q6 |			
Last Name:		First Name:	

4. (15pts) A group of 8 men and 8 women is randomly divided into 2 groups of size 8 each. What is the probability that both groups have the same number of men?

Solution :

$$P(4 \text{ men in each group}) = \frac{\binom{8}{4} \cdot \binom{8}{4}}{\binom{16}{8}}$$

5. (15pts) We have two urns, both containing balls numbered from 1 to 5. We draw one ball from each urn, and note their number. Let E be the event that the two numbers noted are the same.

   (a) (10pts) Let F be the event that the sum of the numbers noted is greater than 5. Are events E and F independent?

   Solution :

$$\begin{aligned} E &= \{(1,1),(2,2),(3,3),(4,4),(5,5)\} \\ F &= \{(1,5),(2,5),(3,5),(4,5),(5,5),(2,4),(3,4),(4,4),(5,4), \\ &\quad (3,3),(4,3),(5,3),(4,2),(5,2),(5,1)\} \end{aligned}$$

   To check whethet E and F are independent, we need to find

$$E \cap F = \{(5,5),(4,4),(3,3)\}$$

$$\mathbb{P}(E \cap F) = \frac{3}{25}, \qquad \mathbb{P}(E) = \frac{5}{25}, \qquad \mathbb{P}(F) = \frac{15}{25}$$

   $\Rightarrow \mathbb{P}(E \cap F) = \mathbb{P}(E) \cdot \mathbb{P}(F)$.
   $\Rightarrow$ Yes, they are independent.

   (b) (5pts) Let G be the event that the sum of the numbers noted is greater than 9. Are events E and G independent?

   Solution :
   We note that

$$\begin{aligned} G &= \{(5,5)\} \\ E \cap G &= \{(5,5)\} \end{aligned}$$

$$\mathbb{P}(E) = \frac{5}{25}, \qquad \mathbb{P}(G) = \frac{1}{25}, \qquad \mathbb{P}(E \cap G) = \frac{1}{25}$$

   $\Rightarrow \mathbb{P}(E \cap G) \neq \mathbb{P}(E) \cdot \mathbb{P}(G)$.

   $\Rightarrow$ No, they are not independent.

Student's Name :________________

6. (20pts) We have two boxes, I and II, with marbles in them. Box I contains 3 red and 2 black marbles, while box II contains 2 red and 8 black marbles. We flip a fair coin: if it comes up heads, we pick a marble from box I, if it comes up tails, we pick a marble from box II. Suppose that we pick just one marble.

   (a) (10pts) What is the probability that the marble we choose is red?
   Solution :
   Define events

$$\begin{aligned} R &= \text{the marble is red,} \\ B_1 &= \text{the marble is from Box I,} \\ B_2 &= \text{the marble is from Box II.} \end{aligned}$$

   Then

$$\begin{aligned} \mathbb{P}(R) &= \mathbb{P}(R|B_1)\mathbb{P}(B_1) + \mathbb{P}(R|B_2)\mathbb{P}(B_2) \\ &= \frac{3}{5}\cdot\frac{1}{2} + \frac{2}{10}\cdot\frac{1}{2} = \frac{2}{5}. \end{aligned}$$

   (b) (10pts) Given that the marble we choose is red, what is the probability of this marble coming from Box I?
   Solution :
   Using Bayes' Theorem, we obtain

$$\mathbb{P}(B_1|R) = \frac{\mathbb{P}(R|B_1)\mathbb{P}(B_1)}{\mathbb{P}(R)} = \frac{\frac{3}{5}\cdot\frac{1}{2}}{\frac{2}{5}} = \frac{3}{4}.$$