January 20, 2017

IŞIK UNIVERSITY, MATH 230 FINAL EXAM

Q1 |	**Q2** |	**Student ID:**	**Row No:**
Q3 |	**Q4** |		
Last Name:		**First Name:**	

1. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, and A_i denote events in a sample space S and let $\mathbb{P}(.)$ denote a probability measure on S.
   (*Note: A statement is assumed to be true if it is true* **in any possible case**, *and it is assumed to be false if it fails in at least one case.*):

   i. For any two events A and B, we always have $\mathbb{P}(A|B) \leq \mathbb{P}(A)$. *T* ⓕ

   ii. If $\mathbb{P}(A) = 0$ then A is the empty event. *T* ⓕ

   iii. The number of r-permutations of n objects is greater then the number of r-combinations of n objects. ⓣ *F*

   iv. For a continuous random variable X and a real number a, we have $\mathbb{P}(X \geq a) = \mathbb{P}(X > a)$. ⓣ *F*

   v. For any random variable X, we have $\mathbb{E}(X^2) - \mathbb{E}(2X) + 1 \geq 0$. ⓣ *F*

   vi. If X and Y are two independent random variables, then the events $(X > 1)$ and $(Y > 1)$ are independent. ⓣ *F*

   vii. Variance of a random variable may equal zero. ⓣ *F*

   viii. A normal random variable has a density curve which is symetric about its mean. ⓣ *F*

   ix. If $\mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) > 1$ then A and B cannot be disjoint. ⓣ *F*

   x. If $A \subset B$ and $\mathbb{P}(A) = 1$ then $\mathbb{P}(B^c) = 0$. ⓣ *F*

2. (14 points) You are visiting a small town in a foreign country with 3 major cell phone carriers, namely A, B and C. It is known that when a tourist dials a number on his/her phone, it will randomly connects to one of these carriers with 40%, 25% and 35% of the time, respectively. It is also known that 90% of the calls through carrier A, 95% of the calls through carrier B and 80% of the calls through carrier C are successful in general.

   i. If you dial a number on your phone in this town, what is the probability that your call will be successful? (Define events and show details of your work to receive full credit.)
   Solution :
   Define events

$$\begin{aligned} S &= \text{call is succesfull,} \\ A &= \text{call goes through carier A,} \\ B &= \text{call goes through carier B,} \\ C &= \text{call goes through carier C.} \end{aligned}$$

   We are given that

$$\mathbb{P}(A) = 0.4, \qquad \mathbb{P}(B) = 0.25, \qquad \mathbb{P}(C) = 0.35,$$

$$\mathbb{P}(S|A) = 0.9, \qquad \mathbb{P}(S|B) = 0.95, \qquad \mathbb{P}(S|C) = 0.8.$$

   Now we need to find $\mathbb{P}(S)$. So

$$\begin{aligned} \mathbb{P}(S) &= \mathbb{P}(S|A)\mathbb{P}(A) + \mathbb{P}(S|B)\mathbb{P}(B) + \mathbb{P}(S|C)\mathbb{P}(C) \\ &= (0.9)(0.4) + (0.95)(0.25) + (0.8)(0.35). \end{aligned}$$

   ii. If you make a successful call on your phone, what is the probability that the call goes through carrier A?
   Solution :
   By Bayes' Theorem,

$$\mathbb{P}(A|S) = \frac{\mathbb{P}(S|A)\mathbb{P}(A)}{\mathbb{P}(S)} = \frac{(0.9)(0.4)}{(0.9)(0.4) + (0.95)(0.25) + (0.8)(0.35)}.$$

Student's Name :____________________

3. (12 points) Consider the following cumulative distribution function of a random variable X:

$$F(x) = \begin{cases} 0 & , x < -2, \\ 0.2 & , -2 \leq x < -1, \\ 0.5 & , -1 \leq x < 1, \\ 1 & , 1 \leq x, \end{cases}$$

i. Find the expectation of X.

Solution :

First, we need to find its PMF

$$p(x) = \begin{cases} 0.2 & , x = -2, \\ 0.3 & , x = -1, \\ 0.5 & , x = 1, \\ 0 & , \text{else}. \end{cases}$$

Then

$$\mathbb{E}(X) = -2(0.2) + (-1)(0.3) + 1(0.5) = -0.2$$

ii. Find the expectation of $X^3 + 1$, that is $\mathbb{E}(X^3 + 1)$.

Solution :

$$\begin{aligned} \mathbb{E}(X^3 + 1) &= \left[(-2)^3 + 1\right](0.2) + \left[(-1)^3 + 1\right](0.3) + \left[(1)^3 + 1\right](0.5) \\ &= -1.4 + 0 + 1 \\ &= -0.4 \end{aligned}$$

4. (14 points)

i. Suppose that a server fails if at least 2 power outages occur in one hour. Moreover, suppose that in this region power outages happen according to a Poisson distribution with an average of 1 in every 5 hours. What is the probability that the server functions properly during a one hour period?
Solution :

Define a Poisson random variable X with parameter $\lambda = \frac{1}{5}$ for one hour period. That is,

$$X = \text{number of power outages in one hour} \sim Poisson(\frac{1}{5})$$

with the PMF

$$p(n) = e^{-1/5}\frac{(1/5)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

Then

$$\mathbb{P}(X < 2) = p(0) + p(1) = e^{-\frac{1}{5}} + e^{-\frac{1}{5}} \cdot \frac{1}{5} = \frac{6e^{-\frac{1}{5}}}{5}.$$

ii. Suppose that people have problems logging onto a particular website with probability 0.2, on each attempt. Assuming the attempts are independent, what is the probability that an individual will be able to log on to the site for the first time on his 4th attempt? (Define a random variable and state its type to receive full credit.)
Solution :

Define

$$X = \text{the number of trials up to the first succesful attempt.}$$

Then $X \sim Geometric(0.2)$. And hence

$$\mathbb{P}(X = 4) = (0.8)^3 \cdot (0.2).$$

Q5 |	Q6 |	Student ID:	Row No:
Q7 |	Q8 |		
Last Name:		First Name:	

5. (12pts) Random variables X and Y have the joint probability density function (PDF)

$$f_{X,Y}(x,y) = \begin{cases} 9x^2y^2 & ,0 \le x \le 1; 0 \le y \le 1 \\ 0 & ,\text{otherwise.} \end{cases}$$

(a) (6pts) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
Solution :

$$f_X(x) = \int_0^1 9x^2y^2dy = 3x^2y^3\Big|_{y=0}^{y=1} = 3x^2, \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 9x^2y^2dx = 3y^2x^3\Big|_{x=0}^{x=1} = 3y^2, \qquad 0 \le y \le 1$$

(b) (6pts) Are X and Y independent random variables?
Solution :
Check if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Since

$$f_X(x)f_Y(y) = \begin{cases} 9x^2y^2 & ,0 \le x \le 1; 0 \le y \le 1 \\ \\ 0 & ,\text{otherwise} \end{cases} = f_{X,Y}(x,y)$$

we conclude that X and Y are independent.

6. (15pts) The joint probability mass function (PMF) of two random variables X and Y is given below:

$p_{X,Y}(x,y)$	$x=-2$	$x=0$	$x=2$
$y=-1$	$1/16$	$1/16$	$1/8$
$y=1$	$1/4$	$1/4$	$1/4$

(a) (5pts) Find the marginal PMF of X, $p_X(x)$, and the marginal PMF of Y, $p_Y(y)$.

Solution :

$$p_X(x) = \begin{cases} \frac{1}{16} + \frac{1}{4} & , x = -2, \\ \frac{1}{16} + \frac{1}{4} & , x = 0, \\ \frac{1}{8} + \frac{1}{4} & , x = 2, \\ 0 & , \text{otherwise.} \end{cases} = \begin{cases} \frac{5}{16} & , x = -2, \\ \frac{5}{16} & , x = 0, \\ \frac{6}{16} & , x = 2, \\ 0 & , \text{otherwise.} \end{cases}$$

and

$$p_Y(y) = \begin{cases} \frac{1}{16} + \frac{1}{16} + \frac{1}{8} & , y = -1 \\ \frac{1}{4} + \frac{1}{4} + \frac{1}{4} & , y = 1 \\ 0 & , \text{otherwise} \end{cases} = \begin{cases} \frac{1}{4} & , y = -1 \\ \frac{3}{4} & , y = 1 \\ 0 & , \text{otherwise} \end{cases}$$

(b) (5pts) Find the conditional PMF of X given $Y=-1$, $p_{X|Y}(x|-1)$.
Solution :

$$p_{X|Y}(x|-1)=\frac{p_{X,Y}(x,-1)}{p_Y(-1)}=\begin{cases}\dfrac{1/16}{1/4} & ,x=-2,\\[2ex] \dfrac{1/16}{1/4} & ,x=0,\\[2ex] \dfrac{1/8}{1/4} & ,x=2,\\[2ex] 0 & ,\text{otherwise.}\end{cases}=\begin{cases}\dfrac{1}{4} & ,x=-2,\\[2ex] \dfrac{1}{4} & ,x=0,\\[2ex] \dfrac{1}{2} & ,x=2,\\[2ex] 0 & ,\text{otherwise.}\end{cases}$$

(c) (5pts) Find $\mathbb{P}(X>Y)$.
Solution :

$$p(X>Y)=p_{X,Y}(0,-1)+p_{X,Y}(2,-1)+p_{X,Y}(2,1)=\frac{1}{16}+\frac{1}{8}+\frac{1}{4}=\frac{7}{16}.$$

Student's Name : ____________________

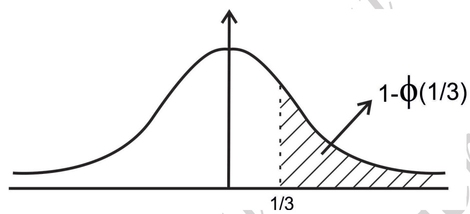
7. (10pts) Let A be a normal random variable with mean 0 and variance 9. Let B be a normal random variable with mean 1 and variance 4. Which probability is larger: $\mathbb{P}(A > 1)$ or $\mathbb{P}(B > 1)$? Show your reasoning for credit.

Solution :

First note that

$$P(A > 1) = P\left(\frac{A}{3} > \frac{1}{3}\right) = 1 - \Phi\left(\frac{1}{3}\right)$$
$$P(B > 1) = P\left(\frac{B-1}{2} > \frac{1-1}{2}\right) = 1 - \Phi(0) = \frac{1}{2}$$

which can be pictured as



Here, $1 - \Phi\left(\frac{1}{3}\right)$ is less than half of the total probability, which is $\frac{1}{2}$.

Hence

$$P(B > 1) > P(A > 1).$$

8. (13pts) Let X be an exponential random variable with $\mathbb{E}[X] = 1$. Define a new random variable $Y = e^X$. Find the pdf of Y, $f_Y(y)$.

Solution :

Since $E[X] = 1$ we know that the parameter is $\lambda = 1$. Then its PDF and CDF are

$$f_X(x) = e^{-x}, \quad x \geq 0 \qquad \text{and} \qquad F_X(x) = 1 - e^{-x}, \quad x \geq 0.$$

By using the given relation,

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X < \ln y) = F_X(\ln y)$$

and so

$$F_Y(y) = 1 - e^{-\ln y} = 1 - \frac{1}{y} \qquad \text{when} \quad \ln(y) \geq 0.$$

Finally,

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{y^2} \quad \text{when} \quad y \geq 1.$$