IŞIK UNIVERSITY, MATH 230 MIDTERM EXAM I

Q1 |	**Q2** |	**Student ID:**	**Row No:**
Q3 |			
Last Name:		**First Name:**	

I pledge my honour that I have not violated the honour code during this examination. Bu sınavda onur yasamızı ihlal etmediğime şerefim üzerine yemin ederim.	**Signature :**

1. (14 points) If A, B and C are independent events, and $\mathbb{P}(A) = 0.3, \mathbb{P}(B) = 0.4, \mathbb{P}(C) = 0.5$, then find the the probability $\mathbb{P}(A \cup B \cup C)$. (**Do NOT use venn diagrams. You won't receive credit unless you show proper mathematical explanation on your paper.**)

Solution :

Since, A, B and C are independent, so are A^c, B^c and C^c. Hence

$$\begin{aligned}\mathbb{P}(A^c \cap B^c \cap C^c) &= \mathbb{P}(A^c)\mathbb{P}(B^c)\mathbb{P}(C^c)\\ &= [1 - \mathbb{P}(A)] \cdot [1 - \mathbb{P}(B)] \cdot [1 - \mathbb{P}(C)]\\ &= (0.7) \cdot (0.6) \cdot (0.5).\end{aligned}$$

Then

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}\left([A^c \cap B^c \cap C^c]^c\right)\\ &= 1 - \mathbb{P}(A^c \cap B^c \cap C^c)\\ &= 1 - (0.7) \cdot (0.6) \cdot (0.5).\end{aligned}$$

2. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, and A_i denote events in a sample space S and let $\mathbb{P}(.)$ denote a probability measure on S.
(*Note: A statement is assumed to be true if it is true* **in any possible case***, and it is assumed to be false if it fails in at least one case.*):

i. If A and B are mutually exclusive then $\mathbb{P}(A \cup B|F) = \mathbb{P}(A|F) + \mathbb{P}(B|F)$ T *F*

ii. If $\mathbb{P}(A) = 1$ then $A = S$. *T* F

iii. $\mathbb{P}(A|B^c) = 1 - \mathbb{P}(A|B)$. *T* F

iv. $\mathbb{P}(A^c|B) = 1 - \mathbb{P}(A|B)$. T *F*

v. If $\mathbb{P}(A|B) = \mathbb{P}(B|A)$ then A and B are equally likely events *T* F

vi. If A and B are independent, then $\mathbb{P}(A|B) = \mathbb{P}(A|B^c)$. T *F*

vii. If A and B are independent then they are mutually exclusive. *T* F

viii. The sum of all coefficients in the expansion of $(x + y)^{15}$ is 2^{15}. T *F*

ix. The coefficient of $x^2y^2z^2$ in the expansion of $(x + y + z)^6$ is 90. T *F*

x. $\mathbb{P}(A) = 0$ then $A = \emptyset$. *T* F

Student's Name :________________

3. (20 points) Deniz is on a vegetarian diet. He is willing to eat his lunch on campus, and hence there are 3 options to him, namely Main Dining Hall, Blue Hall and Red Hall. They prepare random meals everyday. The Main Dining Hall prepares Vegetarian meal 50% of the days, whereas Blue Hall 20% of the days and Red Hall 40% of the days prepare Vegetarian meals.
In order to decide on where to eat, Deniz roll a dice. If the dice shows an even number, he'll choose the Main Dining Hall. If the dice shows a 1 or 3, he'll choose the Red Hall and if the dice shows a 5 he'll choose the Blue Hall. What is the probability that Deniz will have a Vegetarian meal today?

Solution :

Let's define our events:

$$\begin{aligned} M &= \text{he eats in Main Dining Hall,} \\ B &= \text{he eats in Blue Hall,} \\ R &= \text{he eats in Red Hall,} \\ V &= \text{he eats a vegeterian meal.} \end{aligned}$$

We are given

$$\begin{aligned} \mathbb{P}(M) &= \frac{1}{2}, & \mathbb{P}(R) &= \frac{1}{3}, & \mathbb{P}(B) &= \frac{1}{6} \\ \mathbb{P}(V|M) &= 0.5, & \mathbb{P}(V|R) &= 0.4, & \mathbb{P}(V|B) &= 0.2 \end{aligned}$$

Then by the product rule

$$\begin{aligned} \mathbb{P}(V) &= \mathbb{P}(V|M) \cdot \mathbb{P}(M) + \mathbb{P}(V|R) \cdot \mathbb{P}(R) + \mathbb{P}(V|B) \cdot \mathbb{P}(B) \\ &= (0.5) \cdot \frac{1}{2} + (0.4) \cdot \frac{1}{3} + (0.2) \cdot \frac{1}{6}. \end{aligned}$$

Q4 |	Q5 |	Student ID:	Row No:
Q6 |	Q7 |		
Last Name:		First Name:	

4. (16 points)

   i. An urn contains five red and seven blue balls. Suppose that two balls are selected at random and with replacement. Let A and B be the events that the first and the second balls are red, respectively. Are A and B independent events ? Justify your answer by formulating the problem, you do not get any point by saying things like "independent" or "not independent" or "dependent".

   Solution :

   First note that
$$\mathbb{P}(B|A) = \frac{5}{12}.$$

   Next, we write
$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c) \\ &= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^c) \cdot \mathbb{P}(A^c) \\ &= \frac{5}{12} \cdot \frac{5}{12} + \frac{5}{12} \cdot \frac{7}{12} = \frac{5}{12}.\end{aligned}$$

   Since $\mathbb{P}(B|A) = \mathbb{P}(B)$, the events A and B are independent.

ii. Repeat the same problem for the case "without" replacement. Discuss your result.

Solution :

We have

$$\mathbb{P}(B|A) = \frac{4}{11}$$

Moreover,

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B|A)\cdot\mathbb{P}(A) + \mathbb{P}(B|A^c)\cdot\mathbb{P}(A^c)\\ &= \frac{4}{11}\cdot\frac{5}{12} + \frac{5}{11}\cdot\frac{7}{12}\\ &= \frac{5}{12}\end{aligned}$$

Since $\mathbb{P}(B|A) \neq \mathbb{P}(B)$, they are dependent.

5. (12 points) Suppose that four cards are drawn successively from an ordinary deck of 52 cards, with replacement and at random. What is the probability of drawing at least one king ?

Solution :

Define

$$A = \text{drawing at least one king.}$$

Then we have

$$A^c = \text{none of the cards is a king.}$$

So

$$\mathbb{P}(A^c) = \left(\frac{48}{52}\right)^4,$$

and

$$\mathbb{P}(A) = 1 - \left(\frac{48}{52}\right)^4.$$

6. (12 points) A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the box without replacement, if at least one black ball is to be included in the draw?

Solution :

The number of selections with no black balls is

$$\binom{6}{3}.$$

Then the number of selection with at least one black ball is

$$\binom{9}{3} - \binom{6}{3}.$$

Student's Name :____________________

7. (16 points) There are three boxes: B1, B2 and B3. These boxes are filled with balls: 80% of the balls in B1 are white, 90% of the balls in B2 are white and 60% of the balls in B3 are white. The rest of the balls are red. There are the same number of balls in B3 as in B1, whereas B2 contains twice as many of the balls contained in B1. We mix all the balls in a very big box and randomly select a single ball and see that it is white. What is the probability that this ball came from B3 ? Formulate and solve the problem with conditional probability approach.

Solution :

Define events

$$\begin{aligned} W &= \text{selected ball is white,} \\ B_1 &= \text{selected ball is from box B1,} \\ B_2 &= \text{selected ball is from box B2,} \\ B_3 &= \text{selected ball is from box B3.} \end{aligned}$$

We are given that

$$\begin{aligned} \mathbb{P}(W|B_1) &= 0.8, & \mathbb{P}(W|B_2) &= 0.9, & \mathbb{P}(W|B_3) &= 0.6, \\ \mathbb{P}(B_1) &= \frac{1}{4}, & \mathbb{P}(B_2) &= \frac{1}{2}, & \mathbb{P}(B_3) &= \frac{1}{4}. \end{aligned}$$

Hence, by using Bayes' Theorem,

$$\begin{aligned} \mathbb{P}(B_3|W) &= \frac{\mathbb{P}(W|B_3)\cdot\mathbb{P}(B_3)}{\mathbb{P}(W)} \\ &= \frac{\mathbb{P}(W|B_3)\cdot\mathbb{P}(B_3)}{\mathbb{P}(W|B_1)\cdot\mathbb{P}(B_1)+\mathbb{P}(W|B_2)\cdot\mathbb{P}(B_2)+\mathbb{P}(W|B_3)\cdot\mathbb{P}(B_3)} \\ &= \frac{(0.6)\cdot\frac{1}{4}}{(0.8)\cdot\frac{1}{4}+(0.9)\cdot\frac{1}{2}+(0.6)\cdot\frac{1}{4}}. \end{aligned}$$