Işık University **Math 230** **Exam II**

Exam Duration : 1 hr 45 min **Dec. 24, 2012**

Last Name :____________________

First Name :____________________

Student Number :____________________

Section :____________________

Instructor :____________________

Row #:____________________

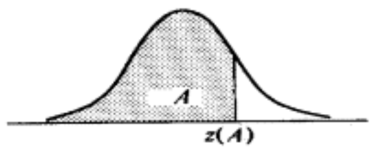
Directions. *Please read each question carefully. Show all work clearly in the space provided. For full credit, solution methods must be complete, logical, and understandable. Answers must be clearly labeled in the spaces provided after each question. Please mark out or fully erase any work that you do not want graded. The point value of each question is indicated before its statement. No books or other references are permitted. Calculators are not allowed. You must show all your work to receive full credit on a problem.*

GRADES			
Q 1	/10	Q 5	/12
Q 2	/18	Q 6	/12
Q 3	/15	Q 7	/12
Q 4	/12	Q 8	/9
Total	/100		

Below, you can find the values of the CDF $\Phi(z)$ of a standard normal (Gaussian) random variable.

Useful Reminder: The density of a Gaussian random variable X with mean μ and variance σ^2 is $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$.

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Student's Name :________________

1. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let X and Y denote random variables, $\mathbb{P}(.)$ denote the probability, $\mathbb{E}(.)$ denote expected value and $Var(.)$ denote variance.
(Note: A statement is assumed to be true if it is true in any case, and it is assumed to be false if it fails in at least one case.):

i. If X is a continuous random variable and $g(x)$ is any function, then $Y = g(X)$ is a continuous random variable. T (F)

ii. If $\mathbb{E}(X) = 0$ then $X = 0$. T F

iii. If $X = 0$ (means X equals zero for any sample point) then $\mathbb{E}(X) = 0$. T F

iv. If X is a continuous random variable and x_0 is any point in the range of X, then $\mathbb{P}(X = x_0) = 0$.. T F

v. If X is a continuous random variable and f_X denotes its PDF, then f_X can take values greater then 1. T F

vi. If X is a discrete random variable and P_X denotes its PMF, then P_X can take values greater then 1. T F

vii. If $P_{X,Y}$ denotes the joint PMF of X and Y and (x_0, y_0) is any point in the range, then $P_{X,Y}(x_0, y_0) \leq P_X(x_0)$. T F

viii. If X is a standard Gaussian, then $\mathbb{P}(X \geq 0) = 1/2$. T F

ix. Let X be a Gaussian with mean μ and variance σ^2, and Y be a Gaussian with mean μ and variance $2\sigma^2$ and f_X, f_Y denote their PDFs. Then the maximum of f_X is greater then the maximum of f_Y. T F

x. For any random variable X, we have $Var(X) \geq 0$. T F

Student's Name : ____________________

2. (18 points)

a.) Let X be a standard Gaussian random variable and Y be a random variable such that $Y = 2X - 1.5$.

i. Find the expected value of Y. [$\mathbb{E}(Y) = ?$]

Solution :

$$\mathbb{E}(Y) = 2\mathbb{E}(X) - 1.5 = 2.0 - 1.5 = -1.5$$

ii. Find the variance of Y. [$Var(Y) = ?$]

Solution :

$$Var(Y) = 4Var(X) = 4 \cdot 1 = 4$$

iii. Find the standard deviation of Y. [$\sigma_Y = ?$]

Solution :

$$\sigma_Y = \sqrt{4} = 2$$

iv. Find the probability $\mathbb{P}(Y \geq 0.5)$?

Solution :

$$Y \sim N(-1.5, 4)$$

$$\begin{aligned} \mathbb{P}(Y \geq 0.5) = \mathbb{P}\left(\frac{Y+1.5}{4} \geq \frac{0.5+1.5}{4}\right) &= \mathbb{P}(Z \geq 0.5) \\ &= 1 - \mathbb{P}(Z < 0.5) \\ &= 1 - \Phi(0.5) \\ &= 1 - 0.5199 \end{aligned}$$

v. Find the probability $\mathbb{P}(Y \leq X)$?

Solution :

$$\mathbb{P}(Y \leq X) = \mathbb{P}(Y - X \leq 0) = \mathbb{P}(X - 1.5 \leq 0) = \mathbb{P}(X \leq 1.5) = \Phi(1.5) = 0.9332$$

b.) What is the value of the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.56} e^{-(x-1)^2/2} dx = ?$$

Solution :

Let $W \sim N(1,1)$. Then this integral equals

$$\begin{aligned} \mathbb{P}(W \leq 1.56) &= \mathbb{P}\left(\frac{W-1}{1} \leq 0.56\right) \\ &= \Phi(0.56) = 0.7123 \end{aligned}$$

Student's Name :____________________

3. (15 points) Let X be a random variable with the PDF

$$f(x) = \begin{cases} c \cdot x^{-4} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$

i. Find the value of c.

Solution :

$$1 = \int_1^\infty cx^{-4}dx = c \cdot \frac{x^{-3}}{-3}\bigg]_1^\infty = \frac{c}{3} \quad \Rightarrow c = 3$$

ii. Find the expected value of X. [$\mathbb{E}(X) =$?]

Solution :

$$\mathbb{E}(X) = \int_1^\infty x \cdot 3x^{-4}dx = \frac{3x^{-2}}{-2}\bigg]_1^\infty = \frac{3}{2}$$

iii. Find the variance of X. [$Var(X) =$?]

Solution :

$$\mathbb{E}(X^2) = \int_1^\infty x^2 \cdot 3x^{-4}dx = 3 \cdot \frac{x^{-1}}{-1}\bigg]_1^\infty = 3$$

$$Var(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

iv. Find the standard deviation of X. [$\sigma_X =$?]

Solution :

$$\sigma_X = \sqrt{\frac{3}{4}}$$

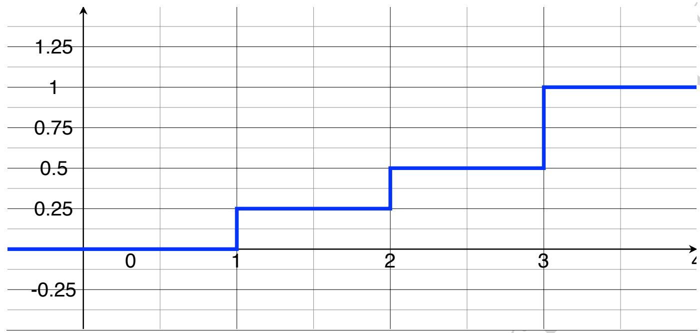
v. Find the probability $\mathbb{P}(0 < X < 2)$?

Solution :

$$\mathbb{P}(0 < X < 2) = \int_1^2 3x^{-4}dx = -x^{-3}\big]_1^2 = 1 - \frac{1}{8} = \frac{7}{8}$$

Student's Name : ____________________

4. (12 points) The CDF of a random variable X is given as follows :



Assume A is the event $A = \{X \geq 2\}$.

i. Write the PMF of X, $P_X(x)$.

Solution :

$$P_X(x) = \begin{cases} 0.25 & , x = 1 \\ 0.25 & , x = 2 \\ 0.5 & , x = 3 \\ 0 & , \text{else} \end{cases}$$

ii. Write the conditional PMF, $P_{X|A}(x)$.

Solution :

$$p_{X|A}(1) = \frac{\mathbb{P}(X = 1, X \geq 2)}{\mathbb{P}(X \geq 2)} = 0$$

$$p_{X|A}(2) = \frac{\mathbb{P}(X = 2, X \geq 2)}{\mathbb{P}(X \geq 2)} = \frac{\mathbb{P}(X = 2)}{\mathbb{P}(X \geq 2)} = \frac{0.25}{0.75} = \frac{1}{3}$$

iii. Find the conditional expectation $\mathbb{E}(X|A)$.

Solution :

$$\begin{aligned}\mathbb{E}(X|A) &= 1 \cdot p_{X|A}(1) + 2 \cdot p_{X|A}(2) + 3 \cdot p_{X|A}(3) \\ &= \frac{2}{3} + 2 = \frac{8}{3}\end{aligned}$$

iv. Find the conditional variance $Var(X|A)$.

Solution :

$$\begin{aligned}\mathbb{E}(X^2|A) &= 1^2 \cdot p_{X|A}(1) + 2^2 \cdot p_{X|A}(2) + 3^2 \cdot p_{X|A}(3) \\ &= 4 \cdot \frac{1}{3} + 9 \cdot \frac{2}{3} = \frac{22}{3}\end{aligned}$$

$$Var(X|A) = \frac{22}{3} - \left(\frac{8}{3}\right)^2 = \frac{2}{9}$$

Q 5	/ 12	Student Number:	Row #:
Q 6	/ 12	First Name:	
Q 7	/ 12	Last Name:	
Q 8	/ 9	Section:	

5. (12 points) The continuous random variable X has the following probability density function

$$f(x) = \begin{cases} ax + bx^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

If the expected value of X is $\mathbb{E}(X) = 0.6$,

i. what are the values of the constants a and b?

Solution :

$$1 = \int_0^1 (ax + bx^2)dx = \left.\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 = \frac{a}{2} + \frac{b}{3}$$

$$0.6 = \int_0^1 (ax^2 + bx^3)dx = \left.\frac{ax^3}{3} + \frac{bx^4}{4}\right]_0^1 = \frac{a}{3} + \frac{b}{4}$$

Solve this system. Then

$$a = 3.6 \qquad \text{and} \qquad b = -2.4$$

ii. What is the probability $\mathbb{P}(X < 1/2)$?

Solution :

$$\mathbb{P}(X < 1/2) = \int_0^{1/2} (3.6x - 2.4x^2)dx = (1.8x^2 - 0.8x^3)_0^{1/2}$$

$$= \frac{1.8}{4} - \frac{0.8}{8} = \frac{2.8}{8} = 0.35$$

iii. What is the variance of X ?

Solution :

$$\mathbb{E}(X^2) = \int_0^1 x(3.6x - 2.4x^2)dx = 1.2x^3 - 0.6x^4\Big]_0^1 = 0.6$$

$$Var(X) = 0.6 - (0.6)^2 = 0.24$$

Student's Name :________________

6. (12 points)

   i. Suppose X, the lifetime of a washing machine (in hours), is a continuous random variable with the following probability density function

$$f(x) = \begin{cases} 10/x^2 & \text{if } x > 10 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the probability $\mathbb{P}(X > 20)$ and find the CDF of X.

   Solution :

$$\mathbb{P}(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = -10x^{-1}\Big]_{20}^{\infty} = \frac{1}{2}$$

   For $x \leq 10, F(x) = 0$

   For $x > 10, F(x) = \int_{10}^{x} \frac{10}{y^2} dy = -\frac{10}{y}\Big]_{10}^{x} = 1 - \frac{10}{x}$

   So

$$F(x) = \begin{cases} 0 & , x \leq 10 \\ 1 - \dfrac{10}{x} & , x > 10 \end{cases}$$

   ii. The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between 0 and 15 minutes. What is the probability that a person waits fewer than 12.5 minutes?

   Solution :

   Let's define

$$X = \text{ time that a person waits for a bus}$$

   Then

$$X \sim Uniform(0, 15)$$

   So

$$\mathbb{P}(X < 12.5) = \int_{0}^{12.5} \frac{1}{15} dx = \frac{12.5}{15}$$

Student's Name :________________

7. (12 points) The time until failure of a vacuum cleaner produced in Kayseri has an exponential distribution and it is known that 50% of vacuum cleaners produced in Kayseri have failed by 1000 hours. (This means, the lifetime of 50% is less than 1000 hours.)

   i. What is the probability that a vacuum cleaner is still working after 5000 hours?

   Solution :

   Let's define

$$X = \text{ life-time of a vacuum cleaner}$$

   Then its PDF is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

   We have

$$0.5 = \mathbb{P}(X < 1000) = \int_0^{1000} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big]_0^{1000} = 1 - e^{-1000\lambda}$$

   So $\lambda = \dfrac{\ln 2}{1000}$. Then

$$\mathbb{P}(X \geq 5000) = \int_{5000}^{\infty} \frac{\ln 2}{1000} \cdot e^{-\frac{\ln 2}{1000}x} dx = -e^{-\frac{\ln 2}{1000}x}\Big]_{5000}^{\infty} = e^{-5\ln 2}$$

   ii. Find the mean (expectation) of the time until failure.

   Solution :

$$\mathbb{E}(X) = \frac{1}{\lambda} = \frac{1000}{\ln 2}$$

Student's Name :________________

8. (9 points) Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} c \cdot |x+y| & \text{if } x = -2, 0, 2 \quad ; \quad y = -1, 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

i. Find the value of c.

Solution :

$$1 = \sum_x \sum_y p_{X,Y}(x,y) = p_{X,Y}(-2,-1) + p_{X,Y}(-2,0) + p_{X,Y}(-2,1) + p_{X,Y}(0,-1)$$

$$+ p_{X,Y}(0,0) + p_{X,Y}(0,1) + p_{X,Y}(2,-1) + p_{X,Y}(2,0) + p_{X,Y}(2,1)$$

$$= 3c + 2c + c + c + c + c + 2c + 3c = 14c$$

$$c = \frac{1}{14}$$

ii. What is the probability $\mathbb{P}(X > Y)$?

Solution :

$$\mathbb{P}(X > Y) = p_{X,Y}(0,-1) + p_{X,Y}(2,-1) + p_{X,Y}(2,0) + p_{X,Y}(2,1)$$

$$= \frac{1}{14} + \frac{1}{14} + \frac{2}{14} + \frac{3}{14} = \frac{7}{14} = \frac{1}{2}$$

iii. What is the probability $\mathbb{P}(X < 1)$?

Solution :

$$\mathbb{P}(X < 1) = p_{X,Y}(-2,-1) + p_{X,Y}(-2,0) + p_{X,Y}(-2,1) + v(0,-1) + p_{X,Y}(0,0) + p_{X,Y}(0,1)$$

$$= \frac{3}{14} + \frac{2}{14} + \frac{1}{14} + \frac{1}{14} + \frac{1}{14}$$

$$= \frac{4}{7}$$