Işık University **Math 230** **Exam I**

Exam Duration : 1 hr 30 min **Nov. 12, 2012**

Last Name :____________________

First Name :____________________

Student Number :____________________

Section : ① ② ③

Instructor : Deniz Karlı ☐ / Sinan Özeren ☐

Row #:____________________

Directions. *Please read each question carefully. Show all work clearly in the space provided. For full credit, solution methods must be complete, logical, and understandable. Answers must be clearly labeled in the spaces provided after each question. Please mark out or fully erase any work that you do not want graded. The point value of each question is indicated before its statement. No books or other references are permitted. Calculators are not allowed. You must show all your work to receive full credit on a problem.*

GRADES			
Q 1	/10	Q 5	/15
Q 2	/12	Q 6	/15
Q 3	/18	Q 7	/15
Q 4	/15		
Total	/100		

Student's Name : ________________

1. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, E, F and A_i denote events in a sample space S and let $P(.)$ denote a probability measure on S.
(*Note: A statement is assumed to be true if it is true* **in any possible case**, *and it is assumed to be false if it fails in at least one case.*):

i. $P(A \cup B) = P(A) + P(B)$. — T (F)

ii. $P(\emptyset) = 0$. — (T) F

iii. $P(\bigcap_{i=1}^{n} A_i^c) = 1 - P(\bigcup_{i=1}^{n} A_i)$. — (T) F

iv. If $P(E \mid S) = P(E)$. — (T) F

v. If $P(A) = 0$ then A is an empty set. — T (F)

vi. If A and B are independent, then $P(A|B) = 1 - P(A^c|B^c)$. — (T) F

vii. If $P(E) > 0$ then $P(E \cap F \mid E) \geq P(E \cap F \mid E \cup F)$. — (T) F

viii. If $P(E) < P(F)$ then $E \subset F$. — T (F)

ix. If A and B are two mutually exclusive events,

then they are independent. — T (F)

x. If X is a discrete random variable,

then $P(X \leq a) = P(X < a)$ for any constant a. — T (F)

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2. (12 points) Determine which of the following functions are PMFs of random variables and which are not. State your answer and your reason clearly in the provided boxes.

(a) $f(x) = \begin{cases} 1/2, & x = 0 \\ 1/3, & x = 1 \\ 1/6, & x = 2 \\ 0, & \text{otherwise} \end{cases}$

Solution :
f is a PMF, since
i.) all values are non-negative and
ii.) $1/2 + 1/3 + 1/6 = 1$

(b) $g(x) = \begin{cases} -1/2, & x = 0 \\ 1/2, & x = 1 \\ 1/2, & x = 2 \\ 0, & \text{otherwise} \end{cases}$

Solution :
g is **NOT** a PMF, since
it returns a negative value;
$g(0) = -1/2$

(c) $h(x) = \begin{cases} 0.2, & x = 0 \\ 0.7, & x = 1 \\ 0.05, & x = 2 \\ 0, & \text{otherwise} \end{cases}$

Solution :
h is **NOT** a PMF, since
its values do not add up to 1;
$0.2 + 0.7 + 0.05 = 0.95 < 1$

Determine which of the following functions are CDFs of random variables and which are not. State your answer and your reason clearly in the provided boxes.

(d) $f(x) = \begin{cases} 1/4, & x = 0 \\ 1/2, & x = 1 \\ 1/4, & x = 2 \end{cases}$

Solution :
f is **NOT** a CDF, since
it is not increasing;
$f(1) > f(2)$

(e) $g(x) = \begin{cases} 1/2, & x = 1 \\ 1/2, & x = 2 \\ 0, & \text{otherwise} \end{cases}$

Solution :
g is **NOT** a CDF, since
it is not increasing;
$g(2) > g(3)$

(f) $h(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \leq x < 10 \\ 1, & x \geq 10 \end{cases}$

Solution :
h is a CDF, since
$h(-\infty) = 0$, $h(\infty) = 1$,
h is increasing
(h is right-contiuous. (*this part is not credited.*))

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3. (18 points)

i. Suppose that, on average, in every three pages of a book there is one mistake. If the number of mistakes on a single page of the book is a Poisson random variable, what is the probability of at least one error on a specific page of the book? What is the expected value of the number of errors?

Solution :

Let X be the r.v. which counts the number of errors. Here we have

$$\lambda = 1/3, \qquad T = 1$$

and hence the parameter

$$\alpha = \lambda \cdot T = 1/3.$$

Now use the PMF of a Poisson R.V. to write

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - P_X(0) = 1 - e^{-1/3}.$$

And the expectation is

$$\mathbb{E}(X) = 1/3.$$

ii. You are sitting in front of your house and observing the plate registrations of the cars that are passing by. You observe a total of 100 cars. If the probability of observing a car with Istanbul plate is 70%, Ankara plate 20% and Izmit plate 10%, what is the probability that out of 100 cars you observe 50 of them have Istanbul plate, 30 of them Ankara plate and 20 of them Izmir plate ?

Solution :

$$\mathbb{P}(\text{Ist. P.}) = 0.7 \qquad \mathbb{P}(\text{Ank. P.}) = 0.2 \qquad \mathbb{P}(\text{Izm. P.}) = 0.1$$

Then

$$\mathbb{P}(50 \text{ Ist. P., } 30 \text{ Ank. P., } 20 \text{ Izm P.}) = \binom{100}{50, 30, 20} (0.7)^{50} (0.2)^{30} (0.1)^{20}$$

iii. A restaurant serves 8 different dishes of fish, 12 of beef, and 10 of poultry. If customers select from these dishes randomly, what is the probability that two of the next four customers order fish entrées?

Solution :

$$\mathbb{P}(2 \text{ Fish, } 2 \text{ Non-Fish}) = \binom{4}{2} \cdot \left[\frac{8}{30}\right]^2 \cdot \left[\frac{22}{30}\right]^2$$

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4. (15 points) Let A, B and C be three events with the following probabilities:

$$P(A) = 0.2 \qquad P(A \cap B) = 0.1 \qquad P(A \cap B \cap C) = 0.08$$
$$P(B) = 0.5 \qquad P(A \cap C) = 0.16$$
$$P(C) = 0.8 \qquad P(B \cap C) = 0.4$$

i. Are the events A and C^c independent?
Solution :

$$P(A)P(C^c) = P(A)(1 - P(C)) = (0.2)(0.2) = 0.04$$

$$P(A \cap C^c) = P(A) - P(A \cap C) = 0.2 - 0.16 = 0.04$$

Yes they are independent.

ii. Are the events B^c and C^c independent?
Solution :

$$P(B^c)P(C^c) = (1 - P(B))(1 - P(C)) = (0.5)(0.2) = 0.1$$

$$\begin{aligned} P(B^c \cap C^c) &= P((B \cup C)^c) = 1 - P(B \cup C) \\ &= 1 - [P(B) + P(C) - P(B \cap C)] \\ &= 1 - [0.5 + 0.8 - 0.4] \\ &= 0.1 \end{aligned}$$

Yes they are independent.

iii. Are the events A, B and C independent?
Solution :

$$P(A \cap B) = 0.1 = (0.2)(0.5) = P(A)P(B)$$

$$P(B \cap C) = 0.4 = (0.5)(0.8) = P(B)P(C)$$

$$P(A \cap C) = 0.16 = (0.2)(0.8) = P(A)P(C)$$

$$P(A \cap B \cap C) = 0.08 = (0.2)(0.5)(0.8) = P(A)P(B)P(C)$$

Yes they are independent.

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5. (15 points) An urn contains 6 green and 2 blue balls. The balls are randomly selected from the urn one after another. If a green ball is selected, it is replaced by another green ball. If a blue ball is selected then it is not replaced and the blue ball is kept outside the urn. The experiment ends when both blue balls are outside the urn. Let X be the random variable recording the number of trial at which the experiment ends. (For example, if the first ball and the second ball selected are blue, the experiment ends and $X = 2$.)

   i. Find $P_X(3)$ where P_X denotes the PMF of the random variable X.
   Solution :
   Let

$$G = \text{the chosen ball is green}$$
$$B = \text{the chosen ball is blue}$$

   Note that $X = 3$ if and only if first two balls were one Green and one Blue and the third ball is Blue. So the event is

$$\{BGB, GBB\}.$$

   Then

$$P_X(3) = \mathbb{P}(\{BGB, GBB\}) = \mathbb{P}(\{BGB\}) + \mathbb{P}(\{GBB\})$$

$$= \frac{2}{8} \cdot \frac{6}{7} \cdot \frac{1}{7} + \frac{6}{8} \cdot \frac{2}{8} \cdot \frac{1}{7}$$

   ii. Find $F_X(3)$ where F_X denotes the CDF of the random variable X.
   Solution :
   Since $P_X(n) = 0$ for $n < 2$, we have

$$F_X(3) = P_X(2) + P_X(3)$$

   where

$$P_X(2) = \mathbb{P}(\{BB\}) = \frac{2}{8} \cdot \frac{1}{7}$$

   Then

$$F_X(3) = \frac{2}{8} \cdot \frac{1}{7} + \frac{2}{8} \cdot \frac{6}{7} \cdot \frac{1}{7} + \frac{6}{8} \cdot \frac{2}{8} \cdot \frac{1}{7}$$

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6. (15 points) In a city in northwestern Turkey, 25% of the cars emit too much pollution (above the safe level). Every year, each car has to be checked for its engine pollution output. The probability that a bad car (a car that procudes excessive pollution) will fail the pollution test is 99%. On the other hand a good car (a car producing little or no pollution) has 17% probability of failing the test. What is the probability that a car failing the test is producing excessive pollution ?

   Solution :

   Let's define the following events first.

$$\begin{aligned} G &= \text{the tested car is a good car.} \\ B &= \text{the tested car is a bad car.} \\ F &= \text{the car fails the test.} \end{aligned}$$

   We are given

$$\begin{aligned} P(B) &= 0.25 \\ P(F|B) &= 0.99 \\ P(F|G) &= 0.17 \end{aligned}$$

   We want to know $P(B|F)$. By Baye's Theorem,

$$\begin{aligned} P(B|F) &= \frac{P(F|B)P(B)}{P(F)} \\ &= \frac{P(F|B)P(B)}{P(F \cap B) + P(F \cap G)} \\ &= \frac{P(F|B)P(B)}{P(F|B)P(B) + P(F|G)P(G)} \\ &= \frac{(0.99)(0.25)}{(0.99)(0.25) + (0.17)(0.75)} \end{aligned}$$

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7. (15 points) A game is played with a pair of dice as follows; one rolls the two dice, and if the sum of the dice S is equal 7 then he/she wins, if S bigger or equal to 10, he/she losses. if S is anything else, the dice is rolled again (that is, the game starts over.). What is the probability of wining?

Solution :

Let's define the following events first.

$$\begin{aligned} W &= \text{win the game} \\ A &= \text{sum of the pair of dice equals 7.} \\ B &= \text{sum of the pair of dice is greater than or equal to 10} \\ C &= \text{sum of the pair of dice rolls is less than 10 and not equal to 7.} \end{aligned}$$

First note that

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(\{(1,6),(2,5),(3,4),(4,3),(5,4),(6,1)\}) = \frac{6}{36} = \frac{1}{6} \\ \mathbb{P}(B) &= \mathbb{P}(\{(6,4),(6,5),(6,6),(5,5),(5,6),(4,6)\}) = \frac{6}{36} = \frac{1}{6} \\ \mathbb{P}(C) &= 1 - \mathbb{P}(A) - \mathbb{P}(B) = 1 - \frac{2}{6} = \frac{2}{3} \end{aligned}$$

Then

$$\mathbb{P}(W) = \mathbb{P}(W \cap A) + \mathbb{P}(W \cap B) + \mathbb{P}(W \cap C)$$

Note that

i. $\mathbb{P}(W \cap A) = \mathbb{P}(W|A)\mathbb{P}(A) = 1 \cdot \mathbb{P}(A) = \frac{1}{6}$

ii. $\mathbb{P}(W \cap B) = 0$ by definition of the game.

iii. $\mathbb{P}(W \cap C) = \mathbb{P}(W|C)\mathbb{P}(C) = \mathbb{P}(W)\mathbb{P}(C) = \frac{2}{3} \cdot \mathbb{P}(W)$. Here $\mathbb{P}(W|C) = \mathbb{P}(W)$ since the game starts over in case of C and the second game is independent of the first game.

Hence

$$\mathbb{P}(W) = \frac{1}{6} + \frac{2}{3} \cdot \mathbb{P}(W)$$

which gives

$$\mathbb{P}(W) = \frac{1}{2}$$