Işık University **Math 230** **Final Exam**

Exam Duration : 2 hr 30 min **Jan. 22, 2013**

Last Name :________________

First Name :________________

Student Number :________________

Section :________________

Instructor :________________

Row #:________________

Directions. *Please read each question carefully. Show all work clearly in the space provided. For full credit, solution methods must be complete, logical, and understandable. Answers must be clearly labeled in the spaces provided after each question. Please mark out or fully erase any work that you do not want graded. The point value of each question is indicated before its statement. No books or other references are permitted. Calculators are not allowed. You must show all your work to receive full credit on a problem.*

GRADES			
Bonus Q.	/10	Q 4	/18
Q 1	/10	Q 5	/9
Q 2	/15	Q 6	/9
Q 3	/24	Q 7	/10
Total	/100 (+10 Bonus)		

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Bonus Question (10 points) In a society, the probability that a random person being a murderer is 0.01. In a murder case you have found a sample of the murderers DNA, and there is a 0.1 probability that a random someones DNA matching this sample. You have found a man whose DNA does match. You also know that if the man is guilty then the DNA matches with probability 0.95. What is the probability that this man is guilty given that his DNA matches? **(Society: "toplum"; murderer: "katil"; match: "eşleşmek, uymak"; sample: "örnek"; guilty: "suçlu")**

Solution :

Let's define our events first.

$$G = \text{ person is guilty of murder,}$$
$$D = \text{person's DNA matches}$$

We have

$$\mathbb{P}(D|G) = 0.95$$
$$\mathbb{P}(D) = 0.1$$
$$\mathbb{P}(G) = 0.01$$

Hence, by using Baye's theorem,

$$\mathbb{P}(G|D) = \frac{\mathbb{P}(D|G)\mathbb{P}(G)}{\mathbb{P}(D)}$$

$$= \frac{(0.95)(0.01)}{0.1}$$

$$= 0.095$$

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1. (10 points) Determine whether the following statements are True or False. Circle **T** or **F**. No explanation is required. Let A, B, E and F denote events in a sample space S and let $\mathbb{P}(.)$ denote a probability measure on S.
(*Note: A statement is assumed to be true if it is true* **in any possible case**, *and it is assumed to be false if it fails in at least one case.*):

i. If X is a continuous random variable and x_0 is any point in the range of X, then $\mathbb{P}(X = x_0) = 0$. (T) F

ii. $\mathbb{P}(\emptyset) = 0$. (T) F

iii. For any random variable X, we have $Var(X) \geq 0$ (T) F

iv. If X is a continuous random variable and f_X denotes its PDF, then f_X can take values greater then 1. (T) F

v. If $\mathbb{P}(A) = 0$ then A is an empty set. T (F)

vi. If A and B are independent, then $P(A|B) = 1 - P(A^c|B^c)$. (T) F

vii. If $P_{X,Y}$ denotes the joint PMF of X and Y and (x_0, y_0) is any point in the range, then $P_{X,Y}(x_0, y_0) > P_X(x_0)$. T (F)

viii. If $P(E) < P(F)$ then $E \subset F$. T (F)

ix. If A and B are two mutually exclusive events,

then they are independent. T (F)

x. If X is a discrete random variable,

then $P(X \leq a) = P(X < a)$ for any constant a. T (F)

2. (15 points) Consider the following joint PMF of X and Y:

$$P_{X,Y}(x,y) = \begin{cases} 0.2 & \text{if } x=0, y=1,2 \\ 0.5 & \text{if } x=3, y=1 \\ 0.1 & \text{if } x=3, y=2 \\ 0 & \text{otherwise.} \end{cases}$$

i. Find the marginal PMF P_X.

Solution :

$$P_X(0) = P_{X,Y}(0,1) + P_{X,Y}(0,2) = 0.4$$
$$P_X(3) = P_{X,Y}(3,1) + P_{X,Y}(3,2) = 0.6$$

$$P_X(x) = \begin{cases} 0.4 & , x=0 \\ 0.6 & , x=3 \\ 0 & , \text{else} \end{cases}$$

ii. Find the marginal PMF P_Y.

Solution :

$$P_Y(1) = P_{X,Y}(0,1) + P_{X,Y}(3,1) = 0.7$$
$$P_Y(2) = P_{X,Y}(0,2) + P_{X,Y}(3,2) = 0.3$$

$$P_Y(x) = \begin{cases} 0.7 & , y=1 \\ 0.3 & , y=2 \\ 0 & , \text{else} \end{cases}$$

iii. Find the expectations $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

Solution :

$$\mathbb{E}(X) = \sum_x x \cdot P_X(x) = 0 \cdot (0.4) + 3 \cdot (0.6) = 1.8$$
$$\mathbb{E}(Y) = \sum_y y \cdot P_Y(y) = 1 \cdot (0.7) + 2 \cdot (0.3) = 1.3$$

iv. Find the correlation between X and Y; $r_{X,Y} = ?$

v. Find the covariance $Cov(X,Y)$.

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3. (27 points) We have a coin and a dice. The coin lands on "Head" with probability 0.2 and lands on "Tail" with probability 0.8. Let X be the random variable such that $X(\text{Head}) = 1$ and $X(\text{Tail}) = 0$. On the other hand, the dice has 6 faces and the random variable Y keeps track of the number showing on the dice. Let Y have the PMF

$$P_Y(y) = \begin{cases} 0.1 & \text{if } y = 1, 2, 3 \\ 0.3 & \text{if } y = 4 \\ c & \text{if } y = 5, 6. \end{cases}$$

We know that X and Y are independent.

i. Find the value of the constant c.

Solution :

$$\begin{aligned} 1 &= \sum_y P_Y(y) = 0.1 + 0.1 + 0.1 + 0.3 + c + c \\ 1 &= 0.6 + 2c \\ c &= 0.2 \end{aligned}$$

ii. Find the expectation $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

Solution :

$$\begin{aligned} \mathbb{E}(X) &= 1 \cdot (0.2) + 0 \cdot (0.8) = 0.2 \\ \mathbb{E}(Y) &= 1 \cdot (0.1) + 2 \cdot (0.1) + 3 \cdot (0.1) + 4 \cdot (0.3) + 5 \cdot (0.2) + 6 \cdot (0.2) = 4 \end{aligned}$$

iii. What is the variance $Var(X)$?

Solution :

$$\begin{aligned} \mathbb{E}(X^2) &= 1^2 \cdot (0.2) + 0^2 \cdot (0.8) = 0.2 \\ Var(X) &= \mathbb{E}(X^2) - \mathbb{E}^2(X) = 0.2 - (0.2)^2 = 0.16 \end{aligned}$$

iv. What is the variance $Var(Y)$?

Solution :

$$\mathbb{E}(Y^2) = 1^2 \cdot (0.1) + 2^2 \cdot (0.1) + 3^2 \cdot (0.1) + 4^2 \cdot (0.3) + 5^2 \cdot (0.2) + 6^2 \cdot (0.2) = 18.4$$
$$Var(Y) = \mathbb{E}(Y^2) - \mathbb{E}^2(Y) = 18.4 - (18.4)^2 = 2.4$$

v. Find the variance $Var(X + Y)$

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vi. What is the value of $\mathbb{E}(2X - Y)$?

Solution :

$$2\mathbb{E}(X) - \mathbb{E}(Y) = 2 \cdot (0.2) - 4 = -3.6$$

vii. Find the conditional probability $\mathbb{P}(Y = 3 | Y \text{ is odd})$.

Solution :

$$= \frac{\mathbb{P}(Y = 3, Y \text{ is odd})}{\mathbb{P}(Y \text{ is odd})} = \frac{\mathbb{P}(Y = 3)}{\mathbb{P}(Y \in \{1, 3, 5\})}$$

$$= \frac{0.1}{0.1 + 0.1 + 0.2} = \frac{1}{4}$$

viii. If $g(x) = 3x^3 - 2$ then find the expected value $\mathbb{E}(g(X))$.

Solution :

$$\sum_x g(x)p_X(x) = g(1)p_X(1) + g(0)p_X(0)$$
$$= 1 \cdot (0.2) + (-2) \cdot (0.8)$$
$$= -1.4$$

ix. Find the moment generating function (MGF) of X.

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4. (18 points) The length of time X, needed to complete a one-hour exam is a random variable with PDF given by

$$f(x) = \begin{cases} a(x^2+x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

i. Calculate the coefficient a.

Solution :

$$1 = \int_0^1 a(x^2+x)dx = a\left(\frac{x^3}{3}+\frac{x^2}{2}\right)_0^1 = a\left(\frac{1}{3}+\frac{1}{2}\right)$$

$$a = \frac{6}{5}$$

ii. Find the cumulative distribution function (CDF).

Solution :

For $x < 0$, $F_X(x) = 0$

For $0 \leq x < 1$, $F_X(x) = \int_0^x \frac{6}{5}(y^2+y)dy = \frac{6}{5}\left(\frac{y^3}{3}+\frac{y^2}{2}\right)_0^x = \frac{6}{5}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)$

For $x \geq 1$, $F_X(x) = 1$

iii. Use the CDF to find $\mathbb{P}(X \leq 0)$ and $\mathbb{P}(X \leq 2)$.

Solution :

$$\mathbb{P}(X \leq 0) = F_X(0) = 0$$
$$\mathbb{P}(X \leq 2) = F_X(2) = 1$$

iv. Find the probability that that a randomly selected student will finish the exam in less than half an hour.

Solution :

$$\mathbb{P}(X < 1/2) = \int_0^{1/2} \frac{6}{5}(x^2 + x)dx = \frac{6}{5}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^{1/2}$$

$$= \frac{6}{5}\left(\frac{1}{24} + \frac{1}{8}\right) = \frac{1}{5}$$

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v. Find the mean time needed to complete a 1 hour exam.

Solution :

$$\mathbb{E}(X) = \int_0^1 x \cdot f(x)dx = \int_0^1 \frac{6}{5}(x^3 + x^2)dx = \frac{6}{5}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)_0^1 = \frac{6}{5}\left(\frac{1}{4} + \frac{1}{3}\right)$$

$$= \frac{6}{5} \cdot \frac{7}{12} = \frac{7}{10}$$

vi. Find the variance and standard deviation of X.

Solution :

$$\mathbb{E}(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 \frac{6}{5}(x^4 + x^3)dx = \frac{6}{5}\left(\frac{x^5}{5} + \frac{x^4}{4}\right)_0^1 = \frac{27}{50}$$

$$Var(X) = \frac{27}{50} - \left(\frac{7}{10}\right)^2 = \frac{5}{100} = \frac{1}{20}$$

$$\lambda_X = \sqrt{\frac{1}{20}}$$

5. (9 points) The lifetime of a lighter follows exponential distribution with a mean of 100 days. Find the probability that the light lighters life

i. exceeds 100 days,

Solution :

Its pdf is

$$f(x) = \begin{cases} \frac{1}{100}e^{-x/100} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

So

$$\mathbb{P}(X > 100) = \int_{100}^{\infty} \frac{1}{100}e^{-x/100}dx = -e^{-x/100}\Big]_{100}^{\infty} = e^{-1}$$

ii. exceeds 400 days,

Solution :

$$\mathbb{P}(X > 400) = \int_{400}^{\infty} \frac{1}{100} e^{-x/100} dx = -e^{-x/100}\Big]_{400}^{\infty} = e^{-4}$$

iii. exceeds 400 days given it exceeds 100 days. (Conditional)

Solution :

$$\mathbb{P}(X > 400 | X > 100) = \frac{\mathbb{P}(X > 400, X > 100)}{\mathbb{P}(x > 100)} = \frac{\mathbb{P}(X > 400)}{\mathbb{P}(X > 100)} = \frac{e^{-4}}{e^{-1}} = e^{-3}$$

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6. (9 points) If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find

i. $\mathbb{P}(2 < X < 5)$,

Solution :

$$\begin{aligned}
&= \mathbb{P}\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right) = \mathbb{P}\left(\frac{-1}{3} < Z < \frac{2}{3}\right) \\
&= \Phi(0.67) - \Phi(-0.33) = \Phi(0.67) - (1 - \Phi(0.33)) \\
&= \Phi(0.67) + \Phi(0.33) - 1 = 0.7486 + 0.6293 - 1
\end{aligned}$$

ii. $\mathbb{P}(X > 0)$,

Solution :

$$\begin{aligned}
&= \mathbb{P}\left(\frac{X-3}{3} > \frac{0-3}{3}\right) = \mathbb{P}(Z > -1) = 1 - \mathbb{P}(Z \leq -1) \\
&= 1 - \Phi(-1) = 1 - (1 - \Phi(1)) = \Phi(1) = 0.8413
\end{aligned}$$

iii. $\mathbb{P}(|X - 3| > 6)$.

Solution :

$$\begin{aligned}
&= \mathbb{P}(-6 < X - 3 > 6) = \mathbb{P}\left(-2 < \frac{X-3}{3} < 2\right) \\
&= \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2)) \\
&= 2\Phi(2) - 1 = 2 \cdot (0.9772) - 1
\end{aligned}$$

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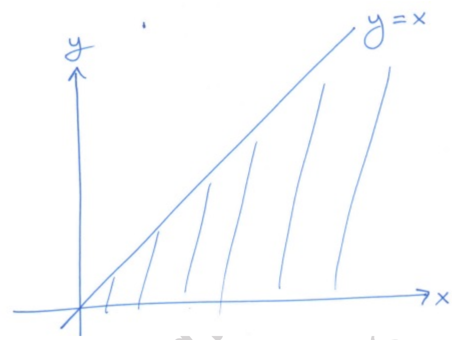
7. (12 points) The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 4e^{-2y} & \text{if } 0 < y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

i. Find the marginal density f_X of X.

Solution :



$$f_X(x) = \int_x^{\infty} 4e^{-2y}dy = -2e^{-2y}\Big]_x^{\infty} = 2e^{-2x}, \quad 0 < x < \infty$$

ii. What is the conditional density $f_{Y|X}(y|X=1)$?

Solution :

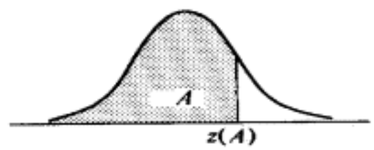
$$f_{Y|X}(y|x=1) = \frac{f_{Y,X}(y,1)}{f_X(1)}$$

$$= \frac{4e^{-2y}}{2e^{-}2} = 2e^{2-2y}, \quad 0 < y < 1$$

Below, you can find the values of the CDF $\Phi(z)$ of a standard normal (Gaussian) random variable.

Useful Reminder: The density of a Gaussian random variable X with mean μ and variance σ^2 is $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$.

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998