4.8 Geometric Random Variable

Suppose independent trials, each $Bernoulli(p)$, are performed until the first success occurs.

$$\begin{array}{ccccccc} 1^{st} \text{ trial fails} & 2^{nd} \text{ trial fails} & \ldots & (n-1)^{th} \text{ trial fails} & n^{th} \text{ trial is successful} & & \\ \uparrow & \uparrow & & \uparrow & \uparrow & & \\ 1-p & 1-p & & 1-p & p & & =(1-p)^{n-1}p \end{array}$$

So if X is the random variable denoting the number of trials up to first success, then

$$\mathbb{P}(X=n)=(1-p)^{n-1}p.$$

This random variable is called a Geometric random variable. We'll write

$$X \sim Geometric(p)$$

where p is probability of success. It takes values 1,2,3,...., and its PMF is

$$p(n)=\begin{cases} (1-p)^{n-1}p & \text{if } n=1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

■ **Example 4.31** An urn contains 10 white 5 black balls. Balls are randomly selected one at a time until a black ball is obtained. Assume each selected ball is replaced before next one is drawn. Let

$$X=\{\text{the number of balls drawn up to the first black ball.}\}$$

So we assume that selecting a black ball is success. In this case

$$p=\text{probability of success}=\frac{5}{15}$$

and

$$1-p=\text{probability of failure}=\frac{10}{15}.$$

Hence the PMF is

$$p(n)=\begin{cases} (\frac{10}{15})^{n-1}(\frac{5}{15}) & \text{if } n=1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then we can answer the following questions.

i. What is the probability that 7 drawn are needed up to the first black ball?

$$p(7)=\left(\frac{10}{15}\right)^6\left(\frac{5}{15}\right)$$

ii. What is the probability that at least 13 drawn are needed up to the first black ball?

$$\begin{aligned}
\mathbb{P}(X \geq 13) &= \sum_{n=13}^{\infty} \left(\frac{10}{15}\right)^{n-1} \left(\frac{5}{15}\right) \\
&= \left(\frac{5}{15}\right) \sum_{n=13}^{\infty} \left(\frac{10}{15}\right)^{n-1} \\
&= \left(\frac{5}{15}\right) \left(\frac{10}{15}\right)^{12} \sum_{n=1}^{\infty} \left(\frac{10}{15}\right)^{n-1} \\
&= \left(\frac{5}{15}\right) \left(\frac{10}{15}\right)^{12} \frac{1}{1-\frac{10}{15}} \\
&= \left(\frac{10}{15}\right)^{12}.
\end{aligned}$$

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Question 11. *If $X \sim Geometric(p)$ then what are the expectation and the variance of X?*

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{n=1}^{\infty} n \cdot p(n) \\
&= \sum_{n=1}^{\infty} (n+1-1) \cdot p(n) \\
&= \sum_{n=1}^{\infty} (n-1) \cdot p(n) + \sum_{n=1}^{\infty} p(n) \\
&= \sum_{n=1}^{\infty} (n-1) \cdot q^{n-1} p + 1 \qquad , \text{ where } q = 1-p \\
&= \sum_{m=0}^{\infty} m \cdot q^{m} p + 1 \\
&= q \sum_{m=0}^{\infty} m \cdot q^{m-1} p + 1 \\
&= q\mathbb{E}(X) + 1.
\end{aligned}$$

Hence we obtain

$$\mathbb{E}(X) = q\mathbb{E}(X) + 1.$$

By solving this equation,

$$\mathbb{E}(X) = \frac{1}{1-q} = \frac{1}{p}.$$

Similarly, one can show that

$$Var(X) = \frac{1}{p^2} - \frac{1}{p}.$$

■ **Example 4.32** Roll a dice until 5 shows up.

$$X = \{\text{the number of trials up to the first 5 shows.}\}$$

Then

$$p = \text{probability of success} = \frac{1}{6}.$$

The PMF is

$$p(n) = \begin{cases} (\frac{5}{6})^{n-1}(\frac{1}{6}) & \text{if } n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that rolling the dice at most 3 times until the dice shows a 5?

$$\begin{aligned} \mathbb{P}(X \le 3) &= p(1) + p(2) + p(3) \\ &= \frac{1}{6} + \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 \\ &= \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2\right] \\ &= \frac{1}{6}\left[\frac{36+30+25}{36}\right] \\ &= \frac{96}{216}. \end{aligned}$$

What is the expectation?

$$\mathbb{E}(X) = \frac{1}{p} = 6.$$

What is the variance?

$$Var(X) = \frac{1}{p^2} - \frac{1}{p} = 36 - 6 = 30.$$

What is the standard deviation?

$$\sigma_X = \sqrt{Var(X)} = \sqrt{30}.$$

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