4.7 Poisson Random Variable

Definition 4.7.1 A random variable X that takes values 0,1,2,3,... is said to be a Poisson random variable with parameter λ if it has the PMF

$$p(n) = \mathbb{P}(X = n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad ,n = 0,1,2,....$$

Poisson random variables have a wide range of applications, mainly since they can be used as an approximation of binomial random variables.

If we consider a binomial random variable with parameters (n, p) where n is large and p is small, then it can be approximated by a Poisson random variable with parameter

$$\lambda = n \cdot p.$$

Some examples:

i. the number of misprints on a page of a book,
ii. the number of people in a community who survive to age 100,
iii. the number of wrong phone numbers dialled in a day,
iv. the number of customers entering a shop on a day.

Example 4.28 Suppose that the number of typographical errors on a single page of a book has a Poisson distribution with $\lambda = 1/2$. What is the probability that there is at least one error on a single page?

$$X = \{\text{the number of errors on a single page}\}$$

$$\begin{aligned}\mathbb{P}(X \geq 1) &= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + ...\\ &= p(1) + p(2) + ...\end{aligned}$$

Since

$$p(n) = \mathbb{P}(X = n) = e^{-1/2}\frac{(1/2)^n}{n!} \quad ,n = 0,1,2,....$$

we have

$$\mathbb{P}(X \geq 1) = e^{-1/2}(1/2) + e^{-1/2}\frac{(1/2)^2}{2!} + e^{-1/2}\frac{(1/2)^3}{3!} + ... = \sum_{n=1}^{\infty} e^{-1/2}\frac{(1/2)^n}{n!}$$

Or

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X = 0) = 1 - e^{-1/2}.$$

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Example 4.29 Suppose that the probability that an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain at most 1 defective item.

$$X = \{\text{the number of defectives}\}$$

Exact Solution: X is a binomial random variable with parameters $n = 10$ and $p = 0.1$. So

$$X \sim Binomial(10, 0.1).$$

Then

$$\begin{aligned}\mathbb{P}(X \leq 1) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1)\\ &= \binom{10}{0}(0.1)^0(0.9)^{10} + \binom{10}{1}(0.1)^1(0.9)^9\\ &= 0.7361.\end{aligned}$$

Approximate Solution: We can approximate X by a Poisson random variable, say Y, with parameter $\lambda = n \cdot p = 10 \cdot (0.1) = 1$. So

$$Y \sim Poisson(1).$$

Then

$$\begin{aligned}\mathbb{P}(X \leq 1) &\approx \mathbb{P}(Y \leq 1) \\ &= \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) \\ &= e^{-1} + e^{-1} \\ &\approx 0.7358.\end{aligned}$$

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Question 10. *If $X \sim Poisson(\lambda)$ what are $\mathbb{E}(X)$ and $Var(X)$?*

Let's calculate the expectation.

$$\begin{aligned}\mathbb{E}(X) &= \sum_{n=0}^{\infty} n \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^n}{(n-1)!} \\ &= \lambda \sum_{n=1}^{\infty} e^{-\lambda} \frac{\lambda^{(n-1)}}{(n-1)!} \\ &= \lambda \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \lambda \sum_{n=0}^{\infty} p(n) \\ &= \lambda.\end{aligned}$$

Next,

$$\begin{aligned}\mathbb{E}(X^2) &= \sum_{n=0}^{\infty} n^2 \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \sum_{n=1}^{\infty} n \cdot e^{-\lambda} \frac{\lambda^n}{(n-1)!} \\ &= \lambda \sum_{n=1}^{\infty} n \cdot e^{-\lambda} \frac{\lambda^{(n-1)}}{(n-1)!} \\ &= \lambda \sum_{n=0}^{\infty} (n+1) \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \lambda \sum_{n=0}^{\infty} (n+1) p(n) \\ &= \lambda \left[\sum_{n=0}^{\infty} n p(n) + \sum_{n=0}^{\infty} p(n) \right] \\ &= \lambda \left[\mathbb{E}(X) + 1 \right] \\ &= \lambda \left[\lambda + 1 \right] \\ &= \lambda^2 + \lambda\end{aligned}$$

Now we can compute the variance of a Poisson random variable.

$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Theorem 4.7.1 If $X \sim Poisson(\lambda)$ then

$$\mathbb{E}(X) = \lambda \quad \text{and} \quad Var(X) = \lambda.$$

■ **Example 4.30** Suppose that earthquakes occur in a certain region with frequency 3 per week on average and occur with probability $1/2$. Find the probability that at least 4 earthquakes occur in the next 5 weeks. Find also expectation and variance.

In order to determine right parameter we need to adapt the time interval according to the question. Since the desired time interval is 5 weeks, we will assume, on average, 15 earthquakes occur in 5 weeks. So we may take

$$n = 15 \quad , \quad p = 1/2$$

Hence

$$\lambda = n \cdot p = 15 \cdot 1/2 = 15/2.$$

So

$$\begin{aligned}\mathbb{P}(X \geq 4) &= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2) - \mathbb{P}(X=3) \\ &= 1 - p(0) - p(1) - p(2) - p(3) \\ &= 1 - e^{-15/2} - e^{-15/2}(15/2) - e^{-15/2}\frac{(15/2)^2}{2!} - e^{-15/2}\frac{(15/2)^3}{3!}\end{aligned}$$

The expectation and the variance are

$$\mathbb{E}(X) = \lambda = 15/2 \quad , \quad Var(X) = \lambda = 15/2.$$

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