4.6 The Bernoulli & Binomial Random Variables

From now on, we start classifying discrete random variables into groups. In this section, we will consider two basic discrete random variables, namely Bernoulli and Binomial random rariables.

4.6.1 Bernoulli Random Variable

Suppose that the outcomes of an experiment can be classified as success or failure. Then a random variable X can be defined as

$$\begin{aligned} X &= 1 \quad &&: \text{when the experiment results in success, and} \\ X &= 0 \quad &&: \text{when the experiment results in failure.} \end{aligned}$$

Then this random variable is called a Bernoulli random variable.

■ **Example 4.21** Flip a coin.

$$\begin{aligned} X &= 1 \quad &&: \text{if the coin shows H, and} \\ X &= 0 \quad &&: \text{if the coin shows T.} \end{aligned}$$

Then X is a Bernoulli random variable. ■

■ **Example 4.22** Roll a dice.

$$\begin{aligned} X &= 1 \quad &&: \text{if the number is even,} \\ X &= 0 \quad &&: \text{if the number is odd.} \end{aligned}$$

Then X is a Bernoulli random variable. ■

A Bernoulli random variable X has the PMF in the form

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

where p is the parameter of Bernoulli random variable and $p \in (0,1)$. Note that no other parameter is needed to describe a Bernoulli random variable. Hence we write

$$X \sim Bernoulli(p)$$

when X is Bernoulli random variable with parameter p.

■ **Example 4.23** Roll a dice.

$$\begin{aligned} X &= 1 \quad &&: \text{if the number is 1,2,3,4,5,} \\ X &= 0 \quad &&: \text{if the number is 6.} \end{aligned}$$

Then X is a Bernoulli random variable with parameter 5/6 which is $\mathbb{P}(X=1)$. We say

$$X \sim Bernoulli(5/6)$$

. Moreover its PMF is given by

$$p(x) = \begin{cases} 5/6 & \text{if } x = 1 \\ 1/6 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

■

4.6.2 Binomial Random Variable

If we repeat a Bernoulli random variable n times, and if we measure the number of success in these n trials, the random variable denoting the number of success is called a Binomial random variable.

■ **Example 4.24** Flip a coin 5 times.

$$X = \{\text{the number of heads}\}$$

Then X is a Binomial random variable, since each flip is a Bernoulli random variable. ■

■ **Example 4.25** Observe the gender of 10 new-born babies.

$$X = \{\text{the number of girls}\}$$

Then X is a Binomial random variable, since each baby is a Bernoulli random variable. ■

Now consider that we have n $Bernoulli(p)$ trials. What is the probability of r success out of these n trials?

$$\underbrace{\_\_\ \ \_\_\ \ \cdots\ \ \_\_}_{\text{n trials}}$$

Consider only the case where the first r trials return success and the last $n-r$ trials are failures. Then this probability is

$$\underbrace{\underline{p}\quad \underline{p}\quad \cdots \quad \underline{p}}_{\text{r trials}}\quad \underbrace{\underline{1-p}\quad \underline{1-p}\quad \cdots\quad \underline{1-p}}_{\text{(n-r) trials}}$$

So this probability equals

$$p^r(1-p)^{n-r}.$$

Remember that we ask for r success in n trials. This means that we need to count the number of possible cases as above. But it's easy to see that all we need to do is to select the r trials when success happens. In how many ways can we choose r trials out of n?

$$\binom{n}{r}$$

Also note that each of these selection has the same probability as above. Hence in total, the probability of r success in n trials is

$$\mathbb{P}(X=r) = \binom{n}{r} p^r(1-p)^{n-r}.$$

Then the PMF of X is

$$p(r) = \begin{cases} \binom{n}{r} p^r(1-p)^{n-r} & \text{if } r = 0,1,2,\ldots,n \\ 0 & \text{otherwise.} \end{cases}$$

A binomial random variable has two parameters; p: probability of single success and n: the number of Bernoulli trials. Hence we denote a binomial random variable X by

$$X \sim Binomial(n,p).$$

■ **Example 4.26** Flip an unfair coin 5 times. The probability of each head is 0.2. What is the probability of 3 heads in 5 trials?

Let X be the random variable counting the number of heads. Then

$$X \sim Binomial(5, 0.2).$$

And the PMF is

$$p(r) = \begin{cases} \binom{5}{r}(0.2)^r(1-0.2)^{5-r} & \text{if } x = 0,1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{5}{r}(0.2)^r(0.8)^{5-r} & \text{if } x = 0,1,2,3,4,5 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$p(3) = \binom{5}{3}(0.2)^3(1-0.2)^2.$$

■

■ **Example 4.27** It is known that screws produced by a certain company will be defective with probability 0.01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Let

$$X = \{\text{the number of defectives in a box}\}$$

Then each screw is either defective or non-defective. Hence the test for each screw is a Bernoulli random variable. In order to test the whole package, one should repeat this experiment 10 times (one for each screw). Moreover at every trial the probability of success (catching a defective screw) is the same as the other, and it equals 0.01. Therefore, X is a binomial random variable.

$$X \sim Binomial(10, 0.01).$$

A package is replaced if at least 2 of the screws are defective. Hence the proportion of the replaced packages is the probability of a package containing at least 2 screws. That is,

$$\begin{aligned} \mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X < 2) = 1 - [\mathbb{P}(X=0) + \mathbb{P}(X=1)] \\ = 1 - \left[\binom{10}{0}(0.01)^0(0.99)^{10} + \binom{10}{1}(0.01)^1(0.99)^9\right] \\ \approx 0.04. \end{aligned}$$

■

Some Properties: Let k be a positive integer, X be a binomial random variable and $p(\cdot)$ be its PMF. Then

$$\begin{aligned} \mathbb{E}(X^k) &= \sum_{r=0}^{n} r^k p(r) \\ &= \sum_{r=0}^{n} r^k \binom{n}{r} p^r (1-p)^{n-r}. \end{aligned}$$

Now use the identity

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

to obtain

$$
\begin{aligned}
\mathbb{E}(X^k) &= \sum_{r=0}^{n} r^k \binom{n}{r} p^r (1-p)^{n-r} \\
&= np \sum_{r=0}^{n} r^{k-1} \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r}
\end{aligned}
$$

Set $s = r-1$ and then

$$
\mathbb{E}(X^k) = np \sum_{s=0}^{n-1} (s+1)^{k-1} \binom{n-1}{s} p^s (1-p)^{n-1-s}.
$$

Note that the term

$$
\binom{n-1}{s} p^s (1-p)^{n-1-s}
$$

corresponds to the PMF of a binomial with n-1 trials and with probability of success p. Let's call this binomial random variable Y. Then the above expression is the expectation

$$
\mathbb{E}(X^k) = np\mathbb{E}\left[(Y+1)^{k-1}\right].
$$

So for $k = 1$, we have

$$
\mathbb{E}(X) = np\mathbb{E}\left[(Y+1)^0\right] = np\mathbb{E}[1] = np.
$$

And for $k = 2$

$$
\mathbb{E}(X^2) = np\mathbb{E}\left[(Y+1)^{2-1}\right] = np\mathbb{E}[Y+1] = np(\mathbb{E}[Y]+1) = np((n-1)p+1).
$$

Hence we can compute the variance by using two equalities above and we obtain

$$
\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= np((n-1)p+1) - (np)^2 \\
&= np(1-p).
\end{aligned}
$$

Proposition 4.6.1 If X is a *Binomial*(n,p) random variable then

i. $\mathbb{E}(X) = np$,
ii. $Var(X) = np(1-p)$.

A sample graph of the PMF of a binomial random variable is as follows: