4.4 Expectation of a Function of a Random Variable

Question 9. *Let X be a discrete random variable and $g(x)$ be a real valued function. What is*

$$\mathbb{E}(g(X)) = ?$$

■ **Example 4.15** Let X be a random variable with PMF

$$p(x) = \begin{cases} 0.2 & \text{if } x = -1 \\ 0.5 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then what is $\mathbb{E}(X^2)$?

Set $g(x) = x^2$ and $Y = X^2$. So basicaly, we are looking for $\mathbb{E}(Y)$. To find this, we need to determine its PMF, say $p_Y(y)$. We note that Y can take values $\{0,1\}$ since it is square of X. Let's determine probabilities of each of these points.

$$\mathbb{P}(Y = 1) = \mathbb{P}(X = 1) + \mathbb{P}(X = -1) = 0.3 + 0.2 = 0.5,$$
$$\mathbb{P}(Y = 0) = \mathbb{P}(X = 0) = 0.5$$

Hence the PMF of Y is

$$p_Y(x) = \begin{cases} 0.5 & \text{if } x = 0 \\ 0.5 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then it is easy to compute the expaectation of Y which was our goal.

$$\mathbb{E}(X^2) = \mathbb{E}(Y) = 0 \cdot (0.5) + 1 \cdot (0.5) = 0.5.$$

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Although the method above can be used to calculate expected value of $g(X)$, there is a more useful interpretation.

Proposition 4.4.1 If X is a discrete random variable that takes values x_i for $i = 1, 2, \ldots$ with probabilities $p(x_i)$ then for any real valued function g,

$$\mathbb{E}(g(X)) = \sum_i g(x_i) \cdot p(x_i).$$

■ **Example 4.16** Let's consider the previous example once again, and compute the desired probability with this new method. Set $g(x) = x^2$.

$$\begin{aligned} \mathbb{E}(X^2) = \mathbb{E}(g(X)) = \sum_i g(x_i) \cdot p(x_i) &= (-1)^2 \cdot p(-1) + (0)^2 \cdot p(0) + (1)^2 \cdot p(1) \\ &= 1 \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = 0.5. \end{aligned}$$

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■ **Example 4.17** Let X be a random variable with PMF

$$p(x) = \begin{cases} 0.2 & \text{if } x = 1 \\ 0.4 & \text{if } x = 3 \\ 0.3 & \text{if } x = 5 \\ 0.1 & \text{if } x = 6 \\ 0 & \text{otherwise} \end{cases}.$$

Find $\mathbb{E}(2X^2-3X)$? By using the proposition above

$$\begin{aligned}\mathbb{E}(2X^2-3X) &= (2\cdot 1^2-3\cdot 1)\cdot p(1)+(2\cdot 3^2-3\cdot 3)\cdot p(3)+(2\cdot 5^2-3\cdot 5)\cdot p(5)+(2\cdot 6^2-3\cdot 6)\cdot p(6)\\ &= -0.2+3.6+10.5+5.4=19.3\end{aligned}$$

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Notation 4.1. *To denote* $\mathbb{E}(X)$, *we will use* μ_X.

Theorem 4.4.2 For any discrete random variable X,

i. $\mathbb{E}(X-\mu_x)=0$,

ii. If a and b are two real numbers then

$$\mathbb{E}(aX+b)=a\mathbb{E}(X)+b.$$

Proof. i. Let $p(x)$ be the PMF of X. Then

$$\mathbb{E}(X-\mu_x)=\sum_x(x-\mu_X)p(x)=\sum_x xp(x)-\sum_x \mu_X p(x)=\mu_x-\mu_x\sum_x p(x)=0$$

since $\sum_x p(x)=1$.

ii. Let a and b be two real numbers. Then

$$\mathbb{E}(aX+b)=\sum_x(ax+b)p(x)=a\sum_x xp(x)+b\sum_x p(x)=a\mu_X+b.$$

■

■ **Example 4.18** If $\mathbb{E}(X)=2$ then $\mathbb{E}\left(\frac{X}{4}-3\right)=\frac{1}{4}\mathbb{E}(X)-3=\frac{2}{4}-3=-5/2$. ■