4.3 Expected Value

Let X be a random variable and $p(x)$ be its PMF. Then the expectation or the expected value or mean of X is defined by

$$\mathbb{E}(X) = \sum_x x \cdot p(x).$$

■ **Example 4.11** Flip a coin.

$$X = \{\text{the number of tails}\}$$

Then

$$S_X = \{0, 1\}$$

and its PMF is

$$p(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We find its expectation by using the definition above. We multiple each possible outcome with its weight (probability) and than sum these values up.

$$\mathbb{E}(X) = 0 \cdot p(0) + 1 \cdot p(1) = 0 \cdot (1/2) + 1 \cdot (1/2) = 1/2.$$

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■ **Example 4.12** Roll a fair dice.

$$X = \{\text{the number that comes up}\}$$

Then

$$S_X = \{1, 2, 3, 4, 5, 6\}$$

and its PMF is

$$p(x) = \begin{cases} 1/6 & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}.$$

Similarly as above

$$\begin{aligned} \mathbb{E}(X) &= 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) + 5 \cdot p(5) + 6 \cdot p(6) + \\ &= 1 \cdot (1/6) + 2 \cdot (1/6) + 3 \cdot (1/6) + 4 \cdot (1/6) + 5 \cdot (1/6) + 6 \cdot (1/6) \\ &= 7/2. \end{aligned}$$

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■ **Example 4.13** A contestant on a quiz show is presented 2 questions, question 1 and question 2, which he is attempt to answer in some order he chooses. If he decides to try question i then he will be allowed to go on to the other question only if he answers question i correctly.

If the contestant receives \$200 for the first and \$100 for the second question and if he knows that he can answer the first question correctly with probability 0.6 and the second question with probability 0.8, which question should he answer first to maximise his earnings?

There are two strategies.

Strategy 1: Start with question 1 then try question 2.

Strategy 2: Start with question 2 then try question 1.

Let's say

$$X = \text{earnings when adapt Strategy 1}$$
$$Y = \text{earnings when adapt Strategy 2}$$

Then we can see that

$$S_X = \{0, 200, 300\} \quad , \quad S_Y = \{0, 100, 300\}.$$

And we have

X		Earnings	Probability
q1 False	:	0	0.4
q1 Correct, q2 False	:	200	$(0.6)\cdot(0.2) = 0.12$
q1 Correct, q2 False	:	300	$(0.6)\cdot(0.8) = 0.48$

and

Y		Earnings	Probability
q2 False	:	0	0.2
q2 Correct, q1 False	:	200	$(0.8)\cdot(0.4) = 0.32$
q2 Correct, q1 False	:	300	$(0.8)\cdot(0.6) = 0.48$

Hence the corresponding PMFs are

$$p_X(x) = \begin{cases} 0.4 & \text{if } x = 0 \\ 0.12 & \text{if } x = 200 \\ 0.48 & \text{if } x = 300 \\ 0 & \text{otherwise} \end{cases},$$

and

$$p_Y(y) = \begin{cases} 0.2 & \text{if } y = 0 \\ 0.32 & \text{if } y = 100 \\ 0.48 & \text{if } y = 300 \\ 0 & \text{otherwise} \end{cases}.$$

Now we can find their expectations:

$$\mathbb{E}(X) = 0\cdot p_X(0) + 200\cdot p_X(200) + 300\cdot p_X(300) = 0\cdot(0.4) + 200\cdot(0.12) + 300\cdot(0.48) = \$168$$

$$\mathbb{E}(Y) = 0\cdot p_Y(0) + 100\cdot p_Y(100) + 300\cdot p_Y(300) = 0\cdot(0.2) + 100\cdot(0.32) + 300\cdot(0.48) = \$176$$

Since the expected earnings is higher with the second strategy, it would be the smart choice to start with question 2. ■

■ **Example 4.14** Let X be a random variable with CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \le x < 1 \\ 3/5 & \text{if } 1 \le x < 2 \\ 4/5 & \text{if } 2 \le x < 3 \\ 9/10 & \text{if } 3 \le x < 3.5 \\ 1 & \text{if } x \ge 3.5 \end{cases}$$

then what is its expectation $\mathbb{E}(X)$?

First we need to write PMF using CDF. We note that the jumps of CDF of a discrete random variable correspond to the values of PMF, and the jump size shows the probability of that point.

$$p(x) = \begin{cases} 1/2 & \text{if } x = 0 \\ 3/5 - 1/2 & \text{if } x = 1 \\ 4/5 - 3/5 & \text{if } x = 2 \\ 9/10 - 4/5 & \text{if } x = 3 \\ 1 - 9/10 & \text{if } x = 3.5 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2 & \text{if } x = 0 \\ 1/10 & \text{if } x = 1 \\ 1/5 & \text{if } x = 2 \\ 1/10 & \text{if } x = 3 \\ 1/10 & \text{if } x = 3.5 \\ 0 & \text{otherwise} \end{cases}.$$

Then the expectation is

$$\begin{aligned}\mathbb{E}(X) &= 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 3.5 \cdot p(3.5) \\ &= 0 \cdot 1/2 + 1 \cdot 1/10 + 2 \cdot 1/5 + 3 \cdot 1/10 + 3.5 \cdot 1/10 \\ &= 23/20.\end{aligned}$$

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