4.2 Discrete Random Variables

Definition 4.2.1 A random variable that can take at most countable number of possible values is said to be discrete. Hence if X is discrete, then

$$S_X = \{s_1, s_2, \ldots\}.$$

Definition 4.2.2 The function $p(\cdot)$ is called the probability mass function of X if it is defined by

$$p(a) = \mathbb{P}(X = a)$$

for any $a \in S_X$. We will write in short PMF for probability mass function.

■ **Example 4.5** Experiment: Flip a fair coin twice.

$$X = \{\text{the number of tails}\}$$

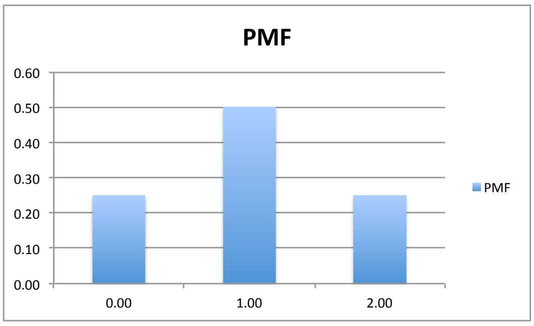
What is the PMF of X? First we see that

$$S_X = \{0, 1, 2\}.$$

The PMF works only on this set. Hence we need to compute 3 corresponding values of $p(x)$.

$$p(0) = \mathbb{P}(X = 0) = \mathbb{P}(\{HH\}) = \frac{1}{4}$$
$$p(1) = \mathbb{P}(X = 1) = \mathbb{P}(\{HT, TH\}) = \frac{2}{4}$$
$$p(2) = \mathbb{P}(X = 2) = \mathbb{P}(\{TT\}) = \frac{1}{4}$$

We use bar graph to graph PMFs of discrete random variables. Here is the graph of PMF,



and the rule of X

$$p(x) = \begin{cases} 1/4 & \text{if } x = 0, 2 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

■

Theorem 4.2.1 Let X be a discrete random variable and $p(\cdot)$ be its PMF. Then for $S_X = \{x_1, x_2, \ldots\}$

i. $p(x_i) \geq 0$ for any $i = 1, 2, \ldots$, and

ii. $\sum_{i=1}^{\infty} p(x_i) = 1$

Proof. i. $p(x_i) = \mathbb{P}(X = x_i) \geq 0$ for any $i = 1, 2, \ldots$.

ii. $\sum_{i=1}^{\infty} p(x_i) = p(x_1) + p(x_1) + \ldots = \mathbb{P}(X = x_1) + \mathbb{P}(X = x_2) + \ldots$ Since the sets $\{X = x_1\}, \{X = x_1\}, \ldots$ are mutually exclusive and their union is the sample space, we have

$$\mathbb{P}(X = x_1) + \mathbb{P}(X = x_2) + \ldots = \mathbb{P}(S) = 1.$$

■

■ **Example 4.6** X is a random variable with PMF

$$p(x) = \begin{cases} c & \text{if } x = 7 \\ 1/3 & \text{if } x = 10 \\ 1/2 & \text{if } x = 100 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of c?

$$1 = \sum_{i=1}^{\infty} p(x_i) = p(7) + p(10) + p(100) = c + 1/3 + 1/2$$

and so $c = 1 - 5/6 = 1/6$. ■

■ **Example 4.7** Which of the following functions are PMF?

a. $p(x) = \begin{cases} -1/2 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$

b. $p(x) = \begin{cases} 1/2 & \text{if } x = 2 \\ 3/2 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

c. $p(x) = \begin{cases} 0.2 & \text{if } x = 10 \\ 0.3 & \text{if } x = 11 \\ 0.5 & \text{if } x = 12 \\ 0 & \text{otherwise} \end{cases}$

Just the last one is PMF. ■

■ **Example 4.8** Let X be a random variable with PMF

$$p(i) = c\frac{\lambda^i}{i!} \quad , i = 0, 1, 2, \ldots$$

where c and λ are constants. Find

$$\mathbb{P}(X \leq 2).$$

(Hint: Use the equality $\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$) First, we need to find c. To do this,

$$1 = \sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} c\frac{\lambda^i}{i!} = c\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = c \cdot e^{\lambda}.$$

Then

$$c = e^{-\lambda},$$

and

$$p(i) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

Hence

$$\begin{aligned}
\mathbb{P}(X \le 2) &= \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) \\
&= p(0) + p(1) + p(2) \\
&= e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2} \\
&= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right).
\end{aligned}$$

■

Definition 4.2.3 Let X be a discrete random variable and $p(x)$ be its PMF. The cumulative distribution function (CDF) of X is defined by

$$F(a) = \sum_{x \le a} p(x).$$

■ **Example 4.9** Experiment: Flip a coin 3 times.

$$X = \{\text{the number of heads}\}$$

Then

$$S_X = \{0, 1, 2, 3\}$$

and

$$\begin{aligned}
p(0) &= \mathbb{P}(X=0) = \mathbb{P}(\{TTT\}) = \frac{1}{8} \\
p(1) &= \mathbb{P}(X=1) = \mathbb{P}(\{TTH, THT, HTT\}) = \frac{3}{8} \\
p(2) &= \mathbb{P}(X=2) = \mathbb{P}(\{THH, HTH, HHT\}) = \frac{3}{8} \\
p(3) &= \mathbb{P}(X=3) = \mathbb{P}(\{HHH\}) = \frac{1}{8}.
\end{aligned}$$

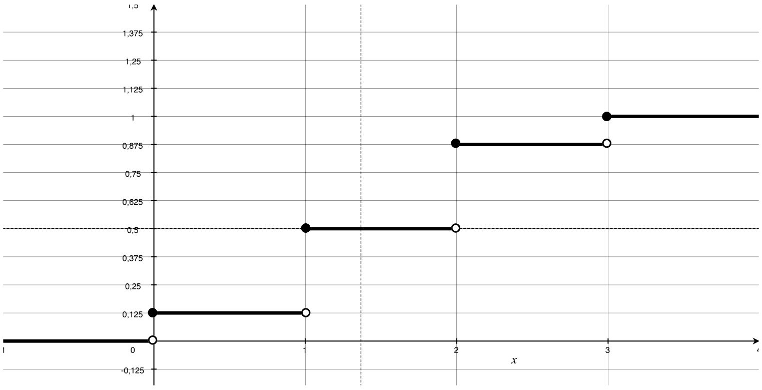
Now we can write the CDF as follows:

$$\begin{aligned}
F(0) &= p(0) = \frac{1}{8} \\
F(1) &= p(0) + p(1) = \frac{4}{8} \\
F(2) &= p(0) + p(1) + p(2) = \frac{7}{8} \\
F(3) &= p(0) + p(1) + p(2) + p(3) = 1.
\end{aligned}$$

What about $F(2.5), F(20), F(-2)$?

$$F(-2) = 0$$
$$F(2.5) = p(0) + p(1) + p(2) = \frac{7}{8}$$
$$F(20) = p(0) + p(1) + p(2) + p(3) = 1.$$

Let's graph the CDF of X.



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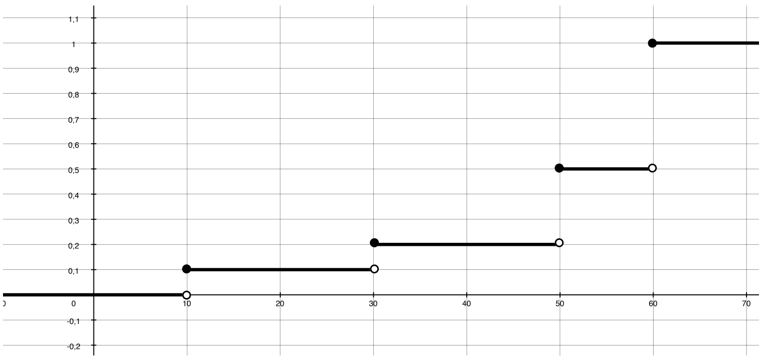
■ **Example 4.10** Let X be a random variable with CDF



Find its PMF.

$$p(x) = \begin{cases} 0.1 & \text{if } x = 10 \\ 0.1 & \text{if } x = 30 \\ 0.3 & \text{if } x = 50 \\ 0.5 & \text{if } x = 60 \\ 0 & \text{otherwise} \end{cases}$$

■

Theorem 4.2.2 If $F(x)$ is a CDF then

i. F is non-decreasing,

ii. $0 \leq F(x) \leq 1$ for any x,

iii. $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$.