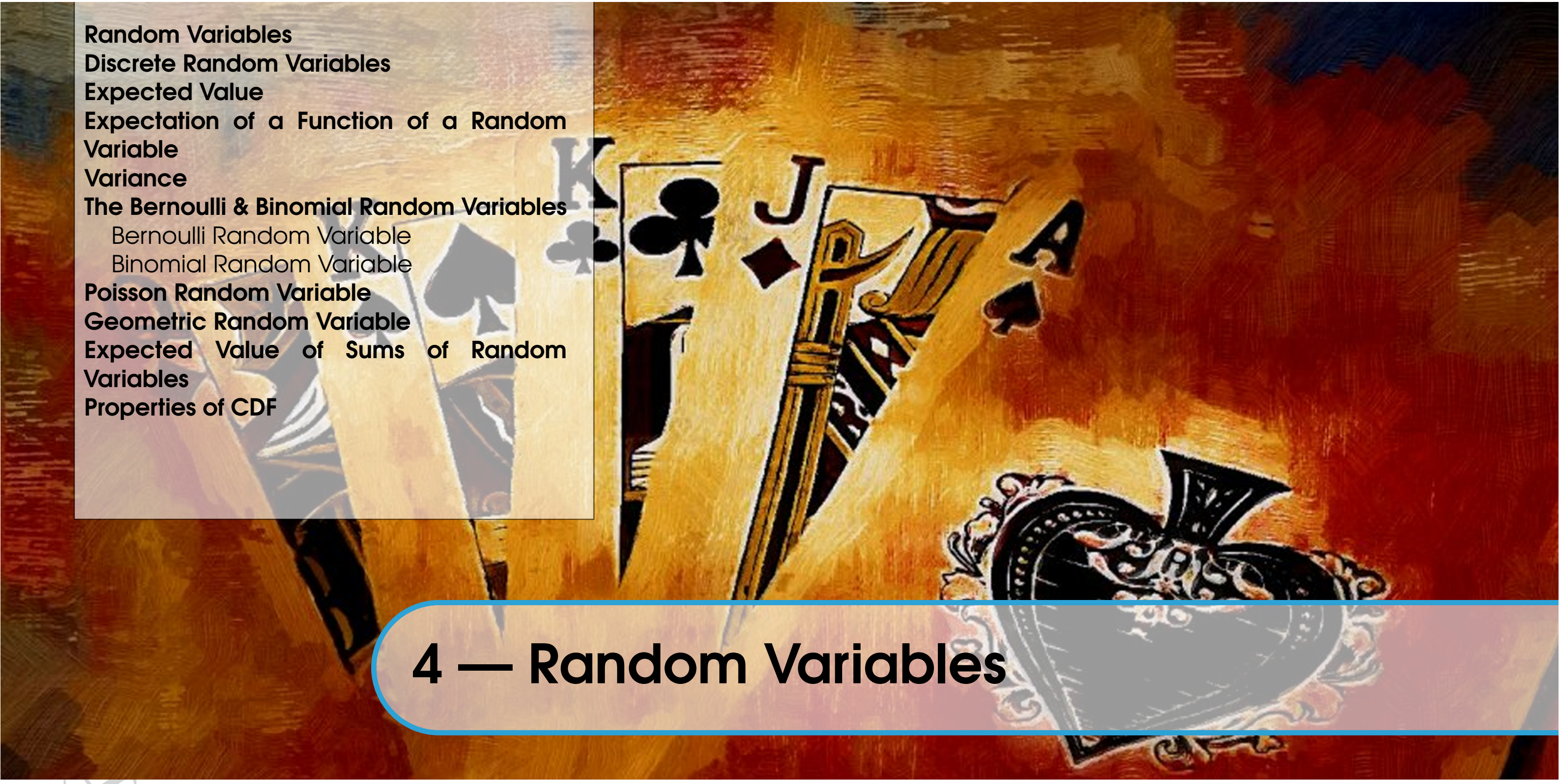


4 — Random Variables

4.1 Random Variables

Let S be a sample space. We call a function

$$X : S \to \mathbb{R}$$

a random variable. We denote random variables by capital letters $X, Y, Z, \ldots$, and the set of all possible values of a random variable X by S_X.

■ **Example 4.1** Experiment: Toss a coin 3 times.

$$X = \{\text{the number of heads}\}$$

After three tosses, there will be either 0, 1, 2 or 3 heads. That means

$$S_X = \{0, 1, 2, 3\}.$$

Note the difference between S_X and S. In this example

$$S = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}.$$

Moreover, we can express the following 4 sets in terms of X:

$$\{X = 0\} = \{TTT\}$$
$$\{X = 1\} = \{TTH, THT, HTT\}$$
$$\{X = 2\} = \{THH, HTH, HHT\}$$
$$\{X = 3\} = \{HHH\}$$

where, for example, the set $\{X = 2\}$ denotes the set of elements from the sample space which correspond to $X = 2$, that is, those elements with two heads. Hence we can easily deduce

$$\mathbb{P}(\{X = 0\}) = 1/8$$
$$\mathbb{P}(\{X = 1\}) = 3/8$$
$$\mathbb{P}(\{X = 2\}) = 3/8$$
$$\mathbb{P}(\{X = 3\}) = 1/8$$

We should note that the sample space is the union of the 4 sets above, which have no intersection. That is,

$$S = \{X = 0\} \cup \{X = 1\} \cup \{X = 2\} \cup \{X = 3\}$$

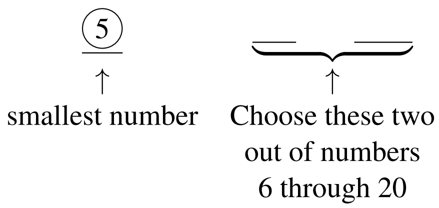
■

R From now on, we will write $\mathbb{P}(X = a)$ instead of $\mathbb{P}(\{X = a\})$.

■ **Example 4.2** 3 balls are selected randomly without replacement from an urn containing 20 balls numbered from 1 through 20.

$$X = \{\text{the smallest of 3 selected numbers}\}$$

What is $\mathbb{P}(X = 5)$?



Hence

$$\mathbb{P}(X = 5) = \frac{\binom{15}{2}}{\binom{20}{3}}.$$

Similarly, for example,

$$\mathbb{P}(X = 18) = \frac{\binom{2}{2}}{\binom{20}{3}},$$

$$\mathbb{P}(X = 12) = \frac{\binom{8}{2}}{\binom{20}{3}},$$

$$\mathbb{P}(X = 19) = 0,$$

$$\mathbb{P}(X \leq 2) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \frac{\binom{19}{2}}{\binom{20}{3}} + \frac{\binom{18}{2}}{\binom{20}{3}}$$

■

■ **Example 4.3** Independent trials consisting of flipping of a coin having probability $1/3$ of coming up heads are continually performed until either a head occurs or a total of 5 flips is made.

$$X = \{\text{the number of times the coin is flipped.}\}$$

Then

$$S_X = \{1,2,3,4,5\}$$

and

$$\mathbb{P}(X=1) = \mathbb{P}(\{H\}) = \frac{1}{3}$$
$$\mathbb{P}(X=2) = \mathbb{P}(\{TH\}) = \frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}$$
$$\mathbb{P}(X=1) = \mathbb{P}(\{TTH\}) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{27}$$
$$\mathbb{P}(X=1) = \mathbb{P}(\{TTTH\}) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{8}{81}$$
$$\mathbb{P}(X=1) = \mathbb{P}(\{TTTTH, TTTTT\}) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{16}{81}$$

■

■ **Example 4.4** Experiment: Picking a random number from the interval $[0,1]$.

$$X = \{\text{the number picked.}\}$$

Some events are $\{X \leq 1/3\} = [0,1/3], \{X > 1\} = \emptyset$ and $\{X = 0.7\} = \{0.7\}$. ■