3.5 P(.|F) is a Probability

Recall the definition of the conditional probability.

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E\cap F)}{\mathbb{P}(F)}.$$

If one fixes the information given, that is the event F, then conditional probability satisfies 3 axioms of the probability measure.

Proposition 3.5.1 Assume E, E_1, E_2,..., F be are events and S is the sample space. Then

i. $0 \leq \mathbb{P}(E|F) \leq 1$

ii. $\mathbb{P}(S|F) = 1$

iii. If $E_1, E_2, ..$ are mutually exclusive events then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i \middle| F\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i|F)$$

Proof. i. By positivity of probability measure,

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E\cap F)}{\mathbb{P}(F)} \geq 0$$

Moreover, since $\mathbb{P}(E\cap F) \leq \mathbb{P}(F)$,

$$0 \leq \mathbb{P}(E|F) \leq 1.$$

ii. By a direct computation,

$$\mathbb{P}(S|F) = \frac{\mathbb{P}(S\cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(F)}{\mathbb{P}(F)} = 1.$$

iii. By using the 3rd axiom of probability,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i \middle| F\right) = \frac{\mathbb{P}((\bigcup_{i=1}^{\infty} E_i)\cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(\bigcup_{i=1}^{\infty}(E_i\cap F))}{\mathbb{P}(F)} = \sum_{i=1}^{\infty} \frac{\mathbb{P}(E_i\cap F)}{\mathbb{P}(F)} = \sum_{i=1}^{\infty} \mathbb{P}(E_i|F).$$

■

We proved all the propositions of the probability measure based on the 3 axioms of probability. Since conditional probability also satisfy these axioms, as a conclusion, conditional probability satisfies these propositions as well. For example,

$$P(A\cup B|F) = \mathbb{P}(A|F) + \mathbb{P}(B|F) - \mathbb{P}(A\cap B|F).$$

(The following example continues from a previous example in Chapter 3.)

■ **Example 3.13** An insurance company classify policy holders into 2 groups: accident-prone and not accident-prone. During any given year, an accident-prone person will have an accident with probability 0.4, whereas the corresponding figure for a not accident-prone person is 0.2. What is the conditional probability that a new policy-holder will have an accident in his/her second year of policy ownership, given that the policyholder has had an accident in the first year?

First, define events

$$\begin{aligned} A &= \text{person is accident-prone} \\ A_1 &= \text{policyholder has an accident in the first year.} \\ A_2 &= \text{policyholder has an accident in the second year.} \end{aligned}$$

We are given that

$$\begin{aligned} \mathbb{P}(A_1|A) &= 0.4 \\ \mathbb{P}(A_2|A) &= 0.4 \\ \mathbb{P}(A_1|A^c) &= 0.2 \\ \mathbb{P}(A_2|A^c) &= 0.2 \end{aligned}$$

From the previous example we also know that

$$\mathbb{P}(A) = 0.3 \quad \text{and} \quad \mathbb{P}(A_1) = 0.26.$$

The question becomes $\mathbb{P}(A_2|A_1) =$? First, conditioning on A,

$$\mathbb{P}(A_2|A_1) = \mathbb{P}(A_2|A_1 \cap A)\mathbb{P}(A|A_1) + \mathbb{P}(A_2|A_1 \cap A^c)\mathbb{P}(A^c|A_1)$$

Here

$$\mathbb{P}(A|A_1) = \frac{\mathbb{P}(A \cap A_1)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(A_1 \cap A)\mathbb{P}(A)}{0.26} = \frac{(0.4)(0.3)}{0.26} = \frac{6}{13}.$$

And so

$$\mathbb{P}(A^c|A_1) = 1 - \mathbb{P}(A|A_1) = 1 - \frac{6}{13} = \frac{7}{13}.$$

Moreover,

$$\mathbb{P}(A_2|A \cap A_1) = 0.4 \quad \text{and} \quad \mathbb{P}(A_2|A^c \cap A_1) = 0.2$$

implies

$$\mathbb{P}(A_2|A_1) = (0.4)\frac{6}{13} + (0.2)\frac{7}{13} \approx 0.29.$$

■

■ **Example 3.14** (Matching Problem)
At a party, n people take off their hats. The hats are then mixed up, and each person randomly selects one. We say that a match occurs if a person selects his/her own hat.

a. What is the probability of no matches?
b. What is the probability of at least 2 matches.

Define the event E by "No match occur". Now, let's denote the probability of no match when there are n people by

$$P_n = \mathbb{P}(E)$$

Let's call the event "the first person selects his/her own hat." as M. Then

$$\begin{aligned} P_n = \mathbb{P}(E) &= \mathbb{P}(E \cap M) + \mathbb{P}(E \cap M^c) \\ &= \underbrace{\mathbb{P}(E|M)}_{=0}\mathbb{P}(M) + \mathbb{P}(E|M^c)\underbrace{\mathbb{P}(M^c)}_{\frac{n-1}{n}} \end{aligned}$$

Hence

$$P_n = \mathbb{P}(E|M^c).\frac{n-1}{n}.$$

Note that $\mathbb{P}(E|M^c)$ is the probability of no match when there are $n-1$ people where one of them has no hat in the collection. The first person picked it in the previous step. So there is an extra person with no hat and extra hat with no owner. In this case,

$$\begin{aligned}\mathbb{P}(E|M^c) &= \begin{pmatrix}\text{no matches and}\\ \text{extra person chooses}\\ \text{extra hat}\end{pmatrix} + \begin{pmatrix}\text{no matches and}\\ \text{extra person does not}\\ \text{choose the extra hat}\end{pmatrix}\\ &= \begin{pmatrix}\text{extra person}\\ \text{chooses the extra hat}\end{pmatrix} \cdot \begin{pmatrix}\text{no match}\\ \text{in the remaining}\\ \text{group of n-2 people}\end{pmatrix} + \begin{pmatrix}\text{no matches and}\\ \text{extra person does not}\\ \text{choose the extra hat}\end{pmatrix}\end{aligned}$$

In the second term, treat the extra hat as if it belongs to the extra person. Then

$$\mathbb{P}(E|M^c) = \frac{1}{n-1}P_{n-2} + P_{n-1}.$$

So

$$P_n = \mathbb{P}(E|M^c)\frac{n-1}{n} = \frac{1}{n}P_{n-2} + \left(1 - \frac{1}{n}\right)P_{n-1}.$$

And

$$P_n - P_{n-1} = -\frac{1}{n}\left(P_{n-1} - P_{n-2}\right)$$

Note that $P_1 = 0$, $P_2 = \frac{1}{2}$.

$$\begin{aligned}P_3 - P_2 &= -\frac{1}{3}(P_2 - P_1) = -\frac{1}{3}\left(\frac{1}{2} - 0\right) = -\frac{1}{3!}\\ \Rightarrow P_3 &= P_2 - \frac{1}{3!} = \frac{1}{2} - \frac{1}{3!}\\ P_4 - P_3 &= -\frac{1}{4}(P_3 - P_2) = -\frac{1}{4}\left(-\frac{1}{3!}\right) = \frac{1}{4!}\\ \Rightarrow P_4 &= \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}.\end{aligned}$$

Hence we obtain

$$P_n = \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \dots + \frac{(-1)^n}{n!}.$$

■