3.4 Independent Events

We have seen that, in general, $\mathbb{P}(A|B) \neq \mathbb{P}(A)$. Note that $\mathbb{P}(A|B)$ is the probability of A given some information whereas $\mathbb{P}(A)$ is the probability where no information is given.

Question 11. *What does* $\mathbb{P}(A|B) = \mathbb{P}(A)$ *mean?*

It means that the probability of A is not effected by the occurrance of B. In this case, we say A and B are independent.

If $\mathbb{P}(A) = \mathbb{P}(A|B)$ then we can write

$$\mathbb{P}(A) = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

and then we obtain

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

which leads us to the definition of independence.

Definition 3.4.1 Two events A and B are called independent if,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

They are dependent if they are not independent.

■ **Example 3.10** Assume that a coin is flipped and a dice is rolled. Then the sample space is

$$S = \{(h,1),(h,2),...,(h,6),(t,1),...,(t,6)\}$$

Next, we define two events

$A =$ coin lands an heads

$B =$ dice shows an even

In this case, we can compute the following probabilities.

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{(h,2),(h,4),(h,6)\}) = \frac{3}{12}$$

$$\mathbb{P}(A) = \mathbb{P}(\{(h,1),(h,2),...,(h,6)\}) = \frac{6}{12}$$

$$\mathbb{P}(B) = \mathbb{P}(\{(h,2),(t,2),(h,4),(t,4),(h,6),(t,6)\}) = \frac{6}{12}$$

So we have

$$\mathbb{P}(A \cap B) = \frac{3}{12} = \frac{6}{12}\frac{6}{12} = \mathbb{P}(A)\mathbb{P}(B)$$

which means that A and B are independent.

If we consider another event C denoting "dice lands on 3" the what can be said about independence of B and C?

$$\mathbb{P}(B|C) = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(C)} = 0$$

$$\mathbb{P}(B) = \frac{6}{12}$$

$$\mathbb{P}(C) = \frac{2}{12}$$

$$\mathbb{P}(B \cap C) \neq \mathbb{P}(B)\mathbb{P}(C)$$

Hence B and C are dependent.

Proposition 3.4.1 If A and B are independent then so are A and B^c, so are A^c and B, so are A^c and B^c.

Proof. Let's prove it for A and B^c.
We know $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ and we want to show $\mathbb{P}(A \cap B^c) = \mathbb{P}(A)\mathbb{P}(B^c)$.
We have

$$\mathbb{P}(A) = \mathbb{P}(A \cap B^c) + \underbrace{\mathbb{P}(A \cap B)}_{= \mathbb{P}(A)\mathbb{P}(B)}$$

since A and B are independent. Then

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A)[1 - \mathbb{P}(B)] = \mathbb{P}(A)\mathbb{P}(B^c).$$

This completes the proof. ■

So if A and B are independent, then we have

$$\mathbb{P}(A|B) = \mathbb{P}(A) \qquad \mathbb{P}(B|A) = \mathbb{P}(B)$$
$$\mathbb{P}(A|B^c) = \mathbb{P}(A) \qquad \mathbb{P}(B^c|A) = \mathbb{P}(B^c)\ldots$$

⊛ ! Warning ! Being mutually exclusive and being independet are different properties! Here is an example to discuss the difference. ■

■ **Example 3.11** Roll 2 dice. Let E,F and G be the events

$E =$ sum equals 7
$F =$ first dice is 4
$G =$ second dice is 3

1. E and F are clearly NOT mutually exclusive. But

$$\mathbb{P}(E \cap F) = \mathbb{P}(\{(4,3)\}) = \frac{1}{36}$$
$$\mathbb{P}(E) = \mathbb{P}(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36}$$
$$\mathbb{P}(F) = \mathbb{P}(\{(4),(4,2),(4,3),(4,4),(4,5),(4,6)\}) = \frac{6}{36}$$

and so

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$$

Hence, E and F are independent.

2. Note that E and F are independent, E and G are independent **but** E and $(F \cap G)$ are not independent. Since

$$\mathbb{P}(E|F \cap G) = 1 \quad \text{but} \quad \mathbb{P}(E) = \frac{1}{6}.$$

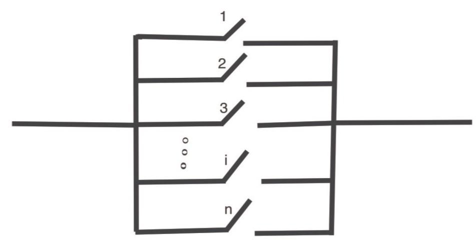
Hence $\mathbb{P}(E|F \cap G) \neq \mathbb{P}(E)$. ■

Definition 3.4.2 Three events A,B and C are independent if

$$1)\,\mathbb{P}(A\cap B\cap C)=\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$
$$2)\,\mathbb{P}(A\cap B)=\mathbb{P}(A)\mathbb{P}(B)$$
$$3)\,\mathbb{P}(A\cap C)=\mathbb{P}(A)\mathbb{P}(C)$$
$$4)\,\mathbb{P}(B\cap C)=\mathbb{P}(B)\mathbb{P}(C)$$

[We can generalize this definition to any number of events in a similar manner.]

■ **Example 3.12** A system composed of n seperate components is said to be a parallel system if it functions when at nleast one of the components functions.



For such a system, if component i, which is independent of the other components, functions with probability p, what is the problem that the system functions?
Let

$$A_i = i^{th} \text{ component functions}$$

and $\mathbb{P}(A_i) = p$. Then

$$\begin{aligned}\mathbb{P}(\text{system functions}) &= \mathbb{P}(\text{at least one component functions})\\ &= 1-\mathbb{P}(\text{none functions})\\ &= 1-\mathbb{P}(A_1^c\cap A_2^c\cap\ldots\cap A_n^c)\\ &= 1-\mathbb{P}(A_1^c)\mathbb{P}(A_2^c)\ldots\mathbb{P}(A_n^c)\\ &= 1-(1-p)^n.\end{aligned}$$

■

<u>Gambler's Ruin Problem:</u>

Two gambler's, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do this until one of them runs out of money. If it is assumed that the successive flips are independent and each flip results in a head with probability $\frac{1}{3}$, what is the probability that A ends up with all the money if he starts with 100 units and B with 50 units?

Let us show the given information on a table.

A	**B**
start with 100 units	start with 50 units
wins 1 unit for each H	wins 1 unit
$\mathbb{P}(\text{win})=\mathbb{P}(H)=\frac{1}{3}$	$\mathbb{P}(\text{win})=\mathbb{P}(T)=\frac{2}{3}$

Total amount of units in this game is 150. Define the event

$$E = A \text{ ends up with all the money (with 150 units).}$$

Let

$$P_i = \text{A ends up with all the money given that he starts with i units.}$$

Then this question can be expressed as

$$P_{100} = ?$$

If

$$H = \text{first flip is heads,} \qquad T = \text{first flip is tails,}$$

then

$$\begin{aligned}\frac{1}{3}P_{100} + \frac{2}{3}P_{100} = P_{100} &= \mathbb{P}(E \cap H) + \mathbb{P}(E \cap T) \\ &= \mathbb{P}(E|H)\mathbb{P}(H) + \mathbb{P}(E|T)\mathbb{P}(T) \\ &= \frac{1}{3}P_{101} + \frac{2}{3}P_{99}\end{aligned}$$

So

$$\frac{1}{3}(P_{101} - P_{100}) = \frac{2}{3}(P_{100} - P_{99})$$

and

$$P_{101} - P_{100} = 2(P_{100} - P_{99}) = 2(2(P_{99} - P_{98})) = 2(2(2(P_{98} - P_{97}))) = ...$$

We conclude

$$\begin{aligned}P_{101} - P_{100} &= 2^{100}(P_1 - P_0), \quad \text{where} \quad P_0 = 0, \\ &= 2^{100}P_1.\end{aligned}$$

Then observe that

$$\begin{aligned}P_{100} - P_{99} &= 2^{99}P_1 \\ P_{99} - P_{98} &= 2^{98}P_1 \\ &. \\ &.. \\ &... \\ P_2 - P_1 &= 2P_1\end{aligned}$$

If we add all of the equations side by side,

$$P_{100} = P_1(1 + 2 + 2^2 + ... + 2^{99})$$

and

$$P_{100} = P_1(2^{100} - 1)$$

To find P_1, note that $P_{150} = 1$. So

$$P_{150} = P_1(2^{150} - 1)$$

and

$$P_1 = \frac{1}{2^{150} - 1}$$

which provides the missing piece. Hence

$$P_{100} = \frac{2^{100} - 1}{2^{150} - 1} \sim \frac{1}{2^{50}}.$$