3.3 Baye's Formula

In the previous section, we discussed the conditional probability of an event in case some piece of information is given. Conditional probability provides an important tool to analyse likelyhood of larger events by dividing them into smaller pieces. In this process, we sometimes need a relation between 2 events when their roles as the main event and the information interchanged. Baye's rule express such a relation which can be obtained through a simple calculation. First, we recall the definition of the conditional probability.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

So we can write

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B)$$

which is referred to as the "Multiplication Rule". Now we can write

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}.$$

Baye's Formula: If $\mathbb{P}(A) \neq 0$ then

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)}.$$

■ **Example 3.4** An insurance company classify people as accident-prone and not-accident-prone. The company's statistics show that an accident-prone person will have an accident sometime within a fixed 1-year period with probability 0.4 whereas this probability decreases to 0.2 for a person who is not accident-prone. If we assume 30% of the population is accident-prone,

a. what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

   As usual, we start by defining the events explicitly.

$$\begin{aligned} A &= \text{person being accident-prone} \\ A^c &= \text{person being not accident-prone} \\ B &= \text{person involving in an accident within a year.} \end{aligned}$$

   We are given that

$$\begin{aligned} \mathbb{P}(A) &= 0.3 \\ \mathbb{P}(A^c) &= 1 - \mathbb{P}(A) = 0.7 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(B|A) &= 0.4 \\ \mathbb{P}(B|A^c) &= 0.2, \end{aligned}$$

   and we are asked for the probability

$$\mathbb{P}(B) = ?$$

By partitioning the event and using the multiplication rule, we obtain

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B\cap A)+\mathbb{P}(B\cap A^c)\\ &= \mathbb{P}(B|A)\mathbb{P}(A)+\mathbb{P}(B|A^c)\mathbb{P}(A^c)\\ &= (0.4)(0.3)+(0.2)(0.7)\\ &= 0.26\end{aligned}$$

b. Suppose that a new policy holder has an accident within a year of purchasing a policy. What is the probability that he/she is accident-prone?
Here note that the question is about $\mathbb{P}(A|B)$, which is in reverse order. Then we can use Baye's rule to relate it to the conditional probability $\mathbb{P}(B|A)$ given in the question.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{(0.4)(0.3)}{0.26} = \frac{12}{26} = \frac{6}{13}.$$

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■ **Example 3.5** In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let 0.6 be the probability that the student knows the answer and 0.4 be the probability that the student guesses. Assume each question has 5 choices. What is the conditional probability that a student knew the answer assuming he answered it correctly?

Define the events as

$K =$ student knows the answer,
$G =$ student guess the answer,
$C =$ student answers correctly.

Then we know

$$\mathbb{P}(K) = 0.6 \qquad \mathbb{P}(C|G) = 1/5$$
$$\mathbb{P}(G) = 0.4 \qquad \mathbb{P}(C|K) = 1$$

The question is about

$$\mathbb{P}(K|C) = ?$$

Then using the Baye's rule

$$\begin{aligned}\mathbb{P}(K|C) &= \frac{\mathbb{P}(K\cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C|K)\mathbb{P}(K)}{\mathbb{P}(C)}\\ &= \frac{\mathbb{P}(C|K)\mathbb{P}(K)}{\mathbb{P}(C\cap K)+\mathbb{P}(C\cap G)}\\ &= \frac{\mathbb{P}(C|K)\mathbb{P}(K)}{\mathbb{P}(C|K)\mathbb{P}(K)+\mathbb{P}(C|G)\mathbb{P}(G)}\\ &= \frac{1.(0.6)}{1.(0.6)+(0.2)(0.4)} = \frac{0.6}{0.68} \cong 0.88.\end{aligned}$$

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■ **Example 3.6** A blood test is 95% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1% of the healthy persons tested. If 0.5% percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Let our events be

$D =$ person has the disease
$P =$ test is positive

Given:

$$\mathbb{P}(P|D) = 0.95, \quad \mathbb{P}(P|D^c) = 0.01, \quad \mathbb{P}(D) = 0.005$$

Question: $\mathbb{P}(D|P) =$? Baye's rule implies that

$$\begin{aligned}
\mathbb{P}(D|P) &= \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P)} \\
&= \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P \cap D) + \mathbb{P}(P \cap D^c)} \\
&= \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P|D)\mathbb{P}(D) + \mathbb{P}(P|D^c)\mathbb{P}(D^c)} \\
&= \frac{(0.95)(0.005)}{(0.95)(0.005) + (0.01)(0.995)} \cong 0.323.
\end{aligned}$$

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ODDS:

Definition 3.3.1 The odds of an event A are defined by;

$$\frac{\mathbb{P}(A)}{\mathbb{P}(A^c)} = \frac{\mathbb{P}(A)}{1-\mathbb{P}(A)}$$

■ **Example 3.7** When you roll a fair dice the odds for the dice shows 1 or 2 is;

$$\frac{\mathbb{P}(\{1,2,3\})}{\mathbb{P}(\{3,4,5,6\})} = \frac{2/6}{4/6} = \frac{1}{2}$$

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⊛ If the odds are $\frac{a}{b}$ we say that the odds in favor of A is a to b, and write a:b.

■ **Example 3.8** In a basketboll game between team A and B, the odds in favor of team A's win are 2:3 . Say

$$A = \text{Team A wins}$$
$$B = \text{Team B wins}$$

Then

$$\frac{\mathbb{P}(A)}{\mathbb{P}(A^c)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{2}{3}$$

Since $\mathbb{P}(A)+\mathbb{P}(B) = 1$ we have

$$\mathbb{P}(B)) + \frac{2}{3}\mathbb{P}(B) = 1$$

and so

$$\mathbb{P}(B) = \frac{3}{5} \quad \& \quad \mathbb{P}(A) = \frac{2}{5}$$

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■ **Example 3.9** An urn contains 2 of type A coins and 1 of type B coin. When a type A coin is flipped, it comes up heads with probability $\frac{1}{4}$, whereas when a type B coin is flipped, it comes up heads with probability $\frac{1}{3}$. A coin is randomly chosen from the urn and flipped. Given that it lands on heads, what is the probability that it was type A?

Events can be defined as

$$A = \text{coin is of type A}$$
$$B = \text{coin is of type B}$$
$$H = \text{coin lands on H.}$$

Given: $\mathbb{P}(A) = \frac{2}{3}, \quad \mathbb{P}(B) = \frac{1}{3}, \quad \mathbb{P}(H|A) = \frac{1}{4}, \quad \mathbb{P}(H|B) = \frac{1}{3}$

Question: $\mathbb{P}(A|H) = ?$

Hence by the Baye's rule,

$$\begin{aligned}
\mathbb{P}(A|H) &= \frac{\mathbb{P}(H|A)\mathbb{P}(A)}{\mathbb{P}(H)} \\
&= \frac{\mathbb{P}(H|A)\mathbb{P}(A)}{\mathbb{P}(H|A)\mathbb{P}(A)+\mathbb{P}(H|B)\mathbb{P}(B)} \\
&= \frac{\frac{1}{4}\cdot\frac{2}{3}}{\frac{1}{4}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{3}} = \frac{3}{5}.
\end{aligned}$$

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