3 — Conditional Probability and Independen

3.2 Conditional Probabilities

In this chapter, we'll discuss the effect of any information presented to the experiment. We notice that we didn't have any knowledge about possible outcomes in the previous chapter. That means, outcomes of an experiment have not been foreseen or eliminated by an outsider. What if somebody tells us some outcomes won't happen?

Let's start with a simple example.

Question 9. *Roll a pair of dice. What is the probability that the sum equals 8?*

This is one of the examples that we discussed several times. So let's write the sample space, the event E and the corresponding probability.

$$\begin{aligned} S &= \{(1,1),(1,2),(1,3),...,(6,6)\} \\ E &= \{(2,6),(3,5),(4,4),(5,3),(6,2)\} \\ \mathbb{P}(E) &= \frac{5}{36}. \end{aligned}$$

Question 10. *What if we have some information on the outcome? Say, we know that the first die show 3, what is the probability that the sum equals 8?*

With this new information, we observe that some of the outcomes given in S are not possible. For example, we won't observe (1,1). Hence we must reconstruct our sample space. Let's say the new sample space is $\mathscr{S}$. The elements in this set are

$$\mathscr{S} = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}.$$

In this sample space, we consider the event

$$\mathscr{E} = \{\text{the sum equals } 8\} = \{(3,5)\}.$$

Then the probability of the sum being 8 is

$$\mathbb{P}(\mathscr{E}) = \frac{|\mathscr{E}|}{|\mathscr{S}|} = \frac{1}{6}$$

which is different from $\mathbb{P}(E)$ above.
Note that if we write

$$F = \{\text{the first die is } 3\} = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$$

then

$$\mathscr{S} = F \quad \text{and} \quad (E) = E \cap F$$

and hence

$$\mathbb{P}(\text{the sum equals 8 given that the first die shows 3}) = \frac{|E \cap F|}{|F|}$$

which can also be represented as

$$\mathbb{P}(\text{the sum equals 8 given that the first die shows 3}) = \frac{|E \cap F|}{|F|} = \frac{\frac{|E\cap F|}{|S|}}{\frac{|F|}{|S|}} = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Notation 3.1. *We will denote the probability of an event A given the information B as* $\mathbb{P}(A|B)$.

For example, the above expression will be written as

$$\mathbb{P}(\text{the sum equals 8 | the first die shows 3}) = \mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Definition 3.2.1 If $\mathbb{P}(E) \neq 0$ then the condition probability of E given that the event F is

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

■ **Example 3.1** A coin is flipped twice. Assuming all outcomes are equally likely, what is the conditional probability that both flips land on heads given that

i. the first flip lands on heads,
ii. at least one flip lands on heads,
iii. both land on tails?

Let's write the sample apace first.

$$S = \{HH, HT, TH, TT\}.$$

i. Denote the event that the first flip is Head by B, and the event that both flips are Heads by A. Then

$$B = \{HH, HT\} \qquad \text{and} \qquad A = \{HH\}.$$

The conditional probability is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\{HH\})}{\mathbb{P}(\{HH, HT\})} = \frac{1/4}{1/2} = \frac{1}{2}.$$

ii. Let C be the event that at least one flip lands on heads. Then

$$C = \{HT, TH, HH\}$$

and hence

$$\mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(\{HH\})}{\mathbb{P}(\{HH, HT, TH\})} = \frac{1/4}{3/4} = \frac{1}{3}.$$

iii. If $D = \{TT\}$ then

$$\mathbb{P}(A|D) = \frac{\mathbb{P}(A \cap D)}{\mathbb{P}(D)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(D)} = 0.$$

■

R The formula in the definition of conditional probability is often written in a product form. We'll note this form here and refer as the multiplication rule as of now.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B).$$

■ **Example 3.2** Celine is undecided as to whether take a French course or a Chemistry course. She estimates that her probability of receiving an A grade would be 1/2 in a French course and 2/3 in a Chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A grade?

Start by defining the events described in the question.

$C =$ Celine takes a Chemistry course,

$F =$ Celine takes a French course,

$A =$ Celine receives an A grade.

We are given that the choice of whether to take French or Chemistry course is based on a fair coin. Hence they are selected with equal probabilities.

$$\mathbb{P}(C) = \frac{1}{2}, \quad \mathbb{P}(F) = 1/2.$$

Moreover, we are give the conditional probabilities

$$\mathbb{P}(A|C) = 2/3, \quad \mathbb{P}(A|F) = 1/2.$$

The question is asking for $\mathbb{P}(A)$. By partitioning the event A and using the multiplication rule,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \cap C) + \mathbb{P}(A \cap F) \\ &= \mathbb{P}(A|C)\mathbb{P}(C) + \mathbb{P}(A|B)\mathbb{P}(B) \\ &= (2/3)(1/2) + (1/2)(1/2) = 7/12.\end{aligned}$$

■

The method used above is common in many cases. If any information about the sample space is given, then we can use this information to partition the sample space and hence to partition any event. For example, if the sample space can be written disjoint union of the events $E_1, E_2, \ldots, E_n$ then for any event A, one can write

$$A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n).$$

Since $E_1, E_2, \ldots, E_n$ are mutually exclusive, it leads to

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \cap E_1) + \mathbb{P}(A \cap E_2) + \ldots + \mathbb{P}(A \cap E_n) \\ &= \mathbb{P}(A|E_1)\mathbb{P}(E_1) + \mathbb{P}(A|E_2)\mathbb{P}(E_2) + \ldots + \mathbb{P}(A|E_n)\mathbb{P}(E_n).\end{aligned}$$

■ **Example 3.3** A total of 46% of voters in a city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24% as Conservatives.

In a recent election 35% of the Independents, 62% of Liberals and 58% of Conservatives voted. A voter is chosen at random. What is the probability that she/he voted?

The first step is to define our events.

$$\begin{aligned} I &= \text{voter is Independent} \\ L &= \text{voter is Liberal,} \\ C &= \text{voter is Conservative,} \\ V &= \text{voter has voted.} \end{aligned}$$

We are given

$$\begin{aligned} \mathbb{P}(I) &= 0.46 & \mathbb{P}(V|I) &= 0.35 \\ \mathbb{P}(L) &= 0.3 & \mathbb{P}(V|L) &= 0.62 \\ \mathbb{P}(C) &= 0.24 & \mathbb{P}(V|C) &= 0.58. \end{aligned}$$

Note that the sample space S can be partitioned into 3 disjoint subsets, namely $S \cap I$, $S \cap L$ and $S \cap C$.So we use this partition for the event V.

$$\begin{aligned} \mathbb{P}(V) &= \mathbb{P}(V \cap I) + \mathbb{P}(V \cap L) + \mathbb{P}(V \cap C) \\ &= \mathbb{P}(V|I)\mathbb{P}(I) + \mathbb{P}(V|L)\mathbb{P}(L) + \mathbb{P}(V|C)\mathbb{P}(C) \\ &= (0.35)(0.46) + (0.62)(0.3) + (0.58)(0.24) = 0.49. \end{aligned}$$

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