2.4 Some Simple Propositions

Theorem 2.4.1

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E).$$

Proof. By Axiom 2 and 3,

$$1 = \mathbb{P}(S) = \mathbb{P}(E \cup E^c) = \mathbb{P}(E) + \mathbb{P}(E^c)$$

since E and E^c are mutually exclusive. Then

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E).$$

■

Theorem 2.4.2 If $E \subset F$ then $\mathbb{P}(E) \leq \mathbb{P}(F)$.

Proof. If $E \subset F$, we can write F as the union of two mutually exclusive sets, namely E and $F - E$. Then by Axiom 3,

$$\mathbb{P}(F) = \mathbb{P}(E) + \mathbb{P}(F - E)$$

Moreover, $\mathbb{P}(F - E)$ is non-negative by Axiom 1. Hence

$$\mathbb{P}(E) \leq \mathbb{P}(F)$$

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■ **Example 2.7** Roll a dice. Then

$$\mathbb{P}(\{2\}) \leq \mathbb{P}(\text{rolling an even number}).$$

■

Theorem 2.4.3 If E and F are two events (not necessarily mutually exclusive) then

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).$$

Proof. Note that E and $F - E$ are mutually exclusive. Hence

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F - E).$$

Moreover, $E \cap F$ and $F - E$ are also mutually exclusive. Then

$$\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(F - E),$$

and

$$\mathbb{P}(F - E) = \mathbb{P}(F) - \mathbb{P}(E \cap F).$$

Hence,

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).$$

■

Example 2.8 Jack is taking two books along on his vacation. With probability 0.5 he'll like the first book, with probability 0.4 he'll like the second book and with probability 0.3 he'll like both books. What is the probability that he likes neither?

Let's denote these two events by
B_1= Jack likes the first book
B_2= Jack likes the second book
The question is $\mathbb{P}\left((B_1 \cup B_2)^c\right)$. We know that

$$\mathbb{P}\left((B_1 \cup B_2)^c\right) = 1 - \mathbb{P}\left(B_1 \cup B_2\right).$$

Moreover,

$$\mathbb{P}\left(B_1 \cup B_2\right) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.5 + 0.4 - 0.3 = 0.6.$$

Hence,

$$\mathbb{P}\left((B_1 \cup B_2)^c\right) = 1 - 0.6 = 0.4.$$

■

Inclusion / Exclusion
We can generalise the inclusion/exclusion property

$$\begin{aligned}\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)\\ - \mathbb{P}(A \cap B)\end{aligned}$$

to more than 2 sets. If A, B, C and D are 4 events then

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)\\ - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C)\\ + \mathbb{P}(A \cap B \cap C).\end{aligned}$$

It is not difficult to see that one can write this property with 4 or more events.

$$\begin{aligned}\mathbb{P}(A \cup B \cup C \cup D) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)\mathbb{P}(D)\\ - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap D) - \mathbb{P}(B \cap C) - \mathbb{P}(B \cap D) - \mathbb{P}(C \cap D)\\ + \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap B \cap D) + \mathbb{P}(A \cap C \cap D) + \mathbb{P}(B \cap C \cap D)\\ - \mathbb{P}(A \cap B \cap C \cap D).\end{aligned}$$