2.3 Axioms of Probability

Let S be a sample space and E be an event. ($E \subset S$.)

Question 8. *How can we define probability of E,*

$$\mathbb{P}(E) = ?$$

■ **Example 2.5** Toss a fair coin. Then the sample space is $S = \{H, T\}$ where H=Heads and T= Tails. Now what is, for example, the probability of H, that is,

$$\mathbb{P}(\{H\}) = ?$$

■

One Approach: relative Frequency

One way to approach this problem is as follows: Repeat this experiment n times and count how many times H occurs in n times. Then look at relative frequency of H:

$$\frac{\text{the number of Heads}}{\text{the number of trials}}.$$

We'll note that as $n \to \infty$, this relative frequency approach a number, in this specific example 0.5. We'll call this number the probability of H, and write

$$\mathbb{P}(\{H\}).$$

Sometimes, we will drop set notation, and write just $\mathbb{P}(H)$, when it is clear that H is an outcome of the experiment.

But note that we assumed that for any repetition of the same experiment, we have

$$\frac{\text{the number of Heads in } n \text{ trials}}{\text{the number of trials}} \to \text{the same number.}$$

This is an extremely strong assumption! (We will call it an axiom.) So in modern probability, we assume another set of axioms to start with.

Axioms of Probability

First, we assume that for any event $E \subset S$, we can assign a number, denoted by $\mathbb{P}(E)$, to the set E. Then

Axiom 1 $0 \leq \mathbb{P}(E) \leq 1$

Axiom 2 $\mathbb{P}(S) = 1$

Axiom 3 For any mutually exclusive events $E_1, E_2, \ldots$ (that is, $E_i \cap E_j = \emptyset$ whenever $i \neq j$), we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

We say $\mathbb{P}(E)$ is the probability of the event E.

By using these axioms, one can prove many fundamental properties of probability measure, some of which will be needed for our studies. Hence we prove a few and leave the rest as exercise.

Theorem 2.3.1

$$\mathbb{P}(\emptyset) = 0.$$

Proof. Let $E_1 = S$ and $E_2 = \emptyset = E_3 = E_4 = \ldots$ Note that these sets are mutually exclusive. Hence by the third Axiom of Probability,

$$\mathbb{P}(S) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i) = \mathbb{P}(S) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \ldots$$

Hence we have

$$0 = \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \ldots$$

Since, by the first Axiom, $\mathbb{P}(\emptyset)$ must be positive, it is zero. ■

Theorem 2.3.2 If A and B are mutually exclusive, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) \cup \mathbb{P}(B).$$

Proof. Let $E_1 = A$, $E_2 = B$ and $\emptyset = E_3 = E_4 = \ldots$ These sets are mutually exclusive. Hence by the third Axiom of Probability,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cup B \cup \emptyset \cup \emptyset \cup \ldots) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(\emptyset) + \mathbb{P}(\emptyset) + \ldots = \mathbb{P}(A) + \mathbb{P}(B)$$

since $\mathbb{P}(\emptyset) = 0$ by Theorem 2.3.1. ■

■ **Example 2.6** Roll a fair dice. then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Since we can write

$$S = \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}$$

and all these sets are mutually exclusive, by Axiom 1 and 3, we have

$$\mathbb{P}(S) = \mathbb{P}(\{1\}) \cup \mathbb{P}(\{2\}) \cup \mathbb{P}(\{3\}) \cup \mathbb{P}(\{4\}) \cup \mathbb{P}(\{5\}) \cup \mathbb{P}(\{6\}).$$

Since the dice is fair, all outcomes are equally-likely, which means all outcomes have the same probability. Then

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = \mathbb{P}(\{5\}) = \mathbb{P}(\{6\}).$$

Hence, using Axion 2, we see that

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \mathbb{P}(\{3\}) = \mathbb{P}(\{4\}) = \mathbb{P}(\{5\}) = \mathbb{P}(\{6\}) = \frac{1}{6}.$$

■