1.5 Multinomial Coefficients

In this section we will generalise the Binomial Theorem to more than 2 variables.

Question 7. *Let's consider n items. We want to divide these objects into r groups of sizes $n_1, n_2, \ldots, n_r$ respectively.*

n - items

n_1 items	n_2 items	...	n_r items

How many such divisions are possible?

We proceed step by step. First, choose n_1 items out of n:

$$\binom{n}{n_1}.$$

Then there are $n - n_1$ items left. Choose n_2 from these remaining ones:

$$\binom{n-n_1}{n_2}.$$

Continue this way until you use all of the items. When we multiply these results, we obtain

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdot \ldots \cdot \binom{n-n_1-\ldots-n_{r-1}}{n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}$$

after cancelations.

Notation 1.1. *If $n_1, n_2, \ldots, n_r$ are positive integers and $n = n_1 + n_2 + \ldots + n_r$ then we write*

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}$$

which represents the number of different divisions of n items into r groups of sizes $n_1, n_2, \ldots, n_r$.

■ **Example 1.12** A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full-time at the station and 3 of the officers on reserve at the station, how many different divisions of 10 officers of the 10 officers into 3 groups are possible?

This is a basic example how multinomial coefficients are useful. First without using multinomials, we can say that we choose 5 patrolling officers out of 10:

$$\binom{10}{5}.$$

Then there are 5 officers left, and we need to choose 2 for working at the station, which is

$$\binom{5}{2}.$$

Finally, the remaining 3 will be on reserve at the station :

$$\binom{3}{3}.$$

So in total

$$\binom{10}{5}\binom{5}{2}\binom{3}{3}$$

which can be simple represented as a multinomial

$$\binom{10}{5,3,2} = \frac{10!}{5!3!2!}.$$

■

R Note that a binomial is a multinomial at the same time.

$$\binom{n}{r} = \binom{n}{r, n-r}.$$

Now we can generalise the Binomial Theorem.

Theorem 1.5.1 — Multinomial Theorem. If n is a positive integer then

$$(x_1 + x_2 + \ldots x_r)^n = \sum_{(n_1, n_2, \ldots, n_r): n_1 + n_2 + \ldots + n_r = n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}.$$

■ **Example 1.13** What is the coefficient of x^4y^3z in the expansion of $(x+y+z)^8$?

By multinomial coefficients,

$$\binom{8}{4,3,1} = \frac{8!}{4!3!1!}.$$

■