1.4 Combinations

Question 4. *What is the difference between the following question?*

i. How many 3-letter words can you write using the letters A, B, C, D, E?

ii. How many 3-element subsets of the set { A, B, C, D, E } can you find?

Recall that the difference between a word and a set is the ordering. Whenever the order of elements of a set interchanges, that yields to the same set. That means by changing the order, one can not obtain a new set. On the contrary, if you change the order of letters in a word, you obtain a new word provided that the letters are different. Hence we need to be careful in computations of different collections since the second experiment above ignores those obtained by interchanging elements.

We already answered the first question before. There are

$$5 \cdot 4 \cdot 3 = 60$$

possible words. How abut the second question? As we mentioned, we need to drop those subsets which are obtained by just reordering of the same elements. Note that whenever we have a collection of 3 elements, we have 3! reordering. Hence each collection repeats itself 3! times. That's why we need to divide the total number of collections by this number repetitions. That is,

$$\frac{5 \cdot 4 \cdot 3}{3!}.$$

Next, we note that the above expression can be written in a nicer form.

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3! \cdot 2 \cdot 1} = \frac{5!}{3! \cdot 2!} = \frac{5!}{3! \cdot (5-3)!}.$$

In general, the number of different groups of r items out of n items (when the order is ignored) is given by

$$\frac{n!}{r! \cdot (n-r)!}.$$

We'll call this as the number of r-combinations of n objects, and denote it by

$$\binom{n}{k} = \frac{n!}{r! \cdot (n-r)!}$$

for $0 \leq r \leq n$. (Read as "n choose r".)

R $0! = 1$

R $\binom{n}{r} = \binom{n}{n-r}$

■ Example 1.7 A committee of 3 people will be formed from a group of 20 people. How many different committees can be formed?

In this example, we note that there is no importance of the order of selection. If you choose person 1 first, then person 2 the second and person 3 the third, then this selection is the same as the selection person 2, person 1, person 3. Even if you change the order of selection, you always choose the same 3 person. Hence they all form the same group. That's why this is a question about number of combination of 3-people out of 20. As we discussed above, that number equals

$$\binom{20}{3} = \frac{20!}{3! \cdot 17!} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2} = 1140.$$

■

■ **Example 1.8** From a group of 5 women and 7 men,

i. how many different committees consisting of 2 women and 3 men can be formed?
As in the previous example, this is a combination problem. Hence we need to consider 2 experiments; choosing 2 women out of 5, and 3 men out of 7:

$$\binom{5}{2} \cdot \binom{7}{3} = 350.$$

ii. what if 2 of the men refuse to be in the same committee?
We can approach this problem in 2 ways.
1^{st} **way:** Let's say these 2 men are called M_1 and M_2. How many of these committees do include these 2 men?

$$\underset{\substack{\uparrow \\ \binom{5}{2}}}{\text{Any 2 women-}} \quad M_1\text{-} \quad M_2\text{-} \quad \underset{\substack{\uparrow \\ \binom{5}{1}}}{\text{Any 2 men except } M_1 \text{ and } M_2}$$

Hence

$$\binom{5}{2} \cdot \binom{5}{1} = 50$$

of these committees include both M_1 and M_2. Since we want to know the number of those which don't include both, we subtract 50 out of the total number of groups, that is, $350 - 50 = 300$.
2^{nd} **way:** We consider 3 possible cases:

Case 1: Committees without any of these 2 men.
Case 2: Committees with M_1 but without M_2.
Case 3: Committees with M_2 but without M_1.

When we count the number of committees in each case, we obtain

Case 1: $\binom{5}{2} \cdot \binom{5}{3}$
Case 2: $\binom{5}{2} \cdot \binom{5}{2}$
Case 3: $\binom{5}{2} \cdot \binom{5}{2}$

and by adding these numbers together, we have the total number

$$\binom{5}{2} \cdot \binom{5}{3} + \binom{5}{2} \cdot \binom{5}{2} + \binom{5}{2} \cdot \binom{5}{2} = 300.$$

■

R It is a good time to note a useful identity on combinations. If n and r are positive integers and $1 \leq r \leq n$ then we can write

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

This is an easy expression to remember. Note that the left hand side of the equation represents the number of different ways to choose r elements out of n objects. This can be done in the following way:
First fix one of the n objects.

$$\underbrace{\star}_{\text{fixed object}} \qquad \underbrace{\circ \quad .. \quad \circ}_{\text{remaining (n-1) objects}}$$

Then the r-groups formed out of n objects either contains this fixed object

$$\underbrace{\star \quad \circ \quad .. \quad \circ}_{\text{r-group including the fixed one and (r-1) chosen from (n-1)}} \qquad \underbrace{\circ \quad .. \quad \circ}_{\text{remaining objects}}$$

or does not contain this fixed one.

$$\underbrace{\circ \quad \circ \quad .. \quad \circ}_{\text{r-group excluding the fixed one, hence chosen from (n-1)}} \qquad \underbrace{\star \quad .. \quad \circ}_{\text{remaining objects including the fixed one}}$$

The number of r-groups containing this fixed object is

$$\binom{n-1}{r-1}$$

since in this case you need to choose the remaining $r-1$ objects out of $n-1$. Similarly, the number of r-groups, not containing the fixed object, is

$$\binom{n-1}{r}$$

, since you need to choose all of r-elements from the remaining $n-1$ objects. So the equality naturally appears.

Binomial Theorem

We begin with a question:

Question 5. *How do you expand* $(x+y)^2$ *or* $(x+y)^3$*?*

Let's expand $(x+y)^2$ first.

$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$

Note that the commutativity of multiplication allows us to write $yx = xy$ and hence we have 2 of the term xy in the expansion. Next,

$$(x+y)^3 = (x+y)(x+y)(x+y) = xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy = 1 \cdot x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + 1 \cdot y^3.$$

Similarly, commutativity leads to $xxy = xyx = yxx$ and $yyx = yxy = xyy$. To determine the coefficient of x^2y, for example, it is enough to count different ways in which one can place the symbol x twice in 3 possible seats.

$$\underline{x}\ \underline{x}\ \underline{\cdot} \qquad \underline{x}\ \underline{\cdot}\ \underline{x} \qquad \underline{\cdot}\ \underline{x}\ \underline{x}$$

Once all x's are places, the remaining seats are reserved for the other symbol y. Hence the number of x^2y terms is the number of different ways of choosing 2 seats out of 3, which is

$$\binom{3}{2}.$$

In a similar way, we can count the number of xy^2. All we need to do is to place the only one symbol of x in one of 3 seats.

$$\underline{x}\ \underline{:}\ \underline{:} \qquad \underline{:}\ \underline{x}\ \underline{:} \qquad \underline{:}\ \underline{:}\ \underline{x}$$

Clearly, this can be done in

$$\binom{3}{1}$$

ways. So we have idea of determining the coefficients in the expansion. Then

$$(x+y)^3 = \binom{3}{3} \cdot x^3 + \binom{3}{2} \cdot x^2 y + \binom{3}{1} \cdot xy^2 + \binom{3}{0} \cdot y^3.$$

We can generalise this idea to any binomial expansion.

Theorem 1.4.1 — Binomial Teorem. If n is a positive integer then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

■ **Example 1.9** What is the coefficient of x^7y^4 in the expansion of $(x+y)^{11}$?
As we deduced above, we need to choose 7 seats for x out of 11 possible seats. That is,

$$\binom{11}{7.}$$

Note that we can also choose seats for y instead of x. In this case, the total number will be

$$\binom{11}{4}$$

which is the same as the previous result. ■

■ **Example 1.10** How many r-element subsets does a n-element set have?
This question is very similar to the one above. To form a r-element subset, we need to choose r elements from the main set. Moreover, it returns the same set if you choose the same elements in a different order. Hence total number of r-element sets is

$$\binom{n}{r.}$$

■

Now let's relate the above example to the Binomial Theorem.

■ **Example 1.11** How many subsets does a n-element set have?
We basically count each possible subset.

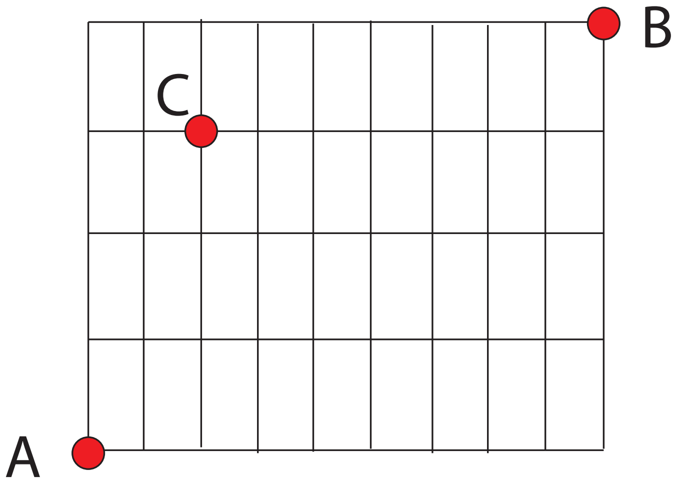
$$\underset{\text{empty set}}{\underset{\uparrow}{\binom{n}{0}}} + \underset{\text{1-element subsets}}{\underset{\uparrow}{\binom{n}{1}}} + \underset{\text{2-element subsets}}{\underset{\uparrow}{\binom{n}{2}}} + \ldots + \underset{n\text{-element subsets.}}{\underset{\uparrow}{\binom{n}{n}}}$$

Note that this is the sum of coefficient in Binomial Theorem, which can be obtained by setting $x = 1$ and $y = 1$. Hence

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r} 1^r 1^{n-r} = (1+1)^n = 2^n.$$

■

Question 6. *Consider the grid below:*



Suppose you want to walk on this grid from the point A to the point B. There are some rules.

- *Each step must be taken from one corner to an adjacent corner.*
- *Each step can be taken one unit up or one unit to the right.*

On how many different paths can you walk form A to B?
Note that there 4 steps up and 9 steps to the right to arrive at the point B. These steps must be fulfilled in any order. If you change the orders of these steps, you obtain a different path. That means the number of paths is basically the number of arrangements of these UP (↑) and RIGHT(→) steps. For example,

$$\uparrow\rightarrow\uparrow\rightarrow\rightarrow\uparrow\uparrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$$

is path. To see the connection, we can assign x to → and y to ↑. Then the path is

$$yxyxxyyxxxxxx.$$

Hence the number of different paths is just the coefficient of x^4y^9, which is

$$\binom{13}{4}.$$

What if you have make a stop at the point C?

Then you need to perform two experiments in order. First, walk from A to C, which can be made in

$$\binom{5}{3}$$

different ways. Then walk from C to A by

$$\binom{8}{1}$$

ways. Hence there are

$$\binom{5}{3}\cdot\binom{8}{1}$$

different path passing through the point C.