1.3 Permutations

Consider the three letters: a, b, and c.

Question 2 (1)**.** *How many 3 letter words (with repetitions) can you write using these letters?(The order of letters does matter, that is, abc≠bac.) The list of of words is:*

$$\left.\begin{array}{c} aaa \\ aab \\ aac \\ aba \\ abb \\ \dots \end{array}\right\} \to 27$$

So we can easily see that

$$\begin{array}{ccccccc} 3 & \cdot & 3 & \cdot & 3 & = & 27 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \{a,b,c\} & & \{a,b,c\} & & \{a,b,c\} & & \#\text{ of words} \end{array}$$

Question 3 (2)**.** *How many 3 letter words (without any repetitions) can you write using these letters? The list of of words is:*

$$\left.\begin{array}{c} abc \\ acb \\ bac \\ bca \\ cab \\ cba \end{array}\right\} \to 6$$

Hence there are

$$\begin{array}{ccccccc} 3 & \cdot & 2 & \cdot & 1 & = & 27 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \{a,b,c\} & & \text{one of two} & & \text{the last} & & \#\text{ of words} \\ \{a,b,c\} & & \text{remaining} & & \text{remaining} & & \\ \{a,b,c\} & & \text{letters} & & \text{letter} & & \end{array}$$

So if we have n objects, there are

$$n \cdot (n-1) \cdot (n-2) \dots 2 \cdot 1 = n!$$

different permutations of them.

■ **Example 1.2** A class consists of 6 men and 4 women. An examination is given and the students are ranked according to their performances. Assume that no two students obtain the same score.

i. How many different ranking are possible?

   It's just the number of arrangements of 10 people. So it is

$$10!$$

ii. If men are ranked among themselves and women are ranked among themselves, how many different rankings are possible?

$$\begin{array}{ccc} 6! & \cdot & 4! \\ \uparrow & & \uparrow \\ \text{6 Men} & & \text{4 Women} \end{array}$$

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■ **Example 1.3** You'll place 10 books on a shelf. Of these, 4 are Math., 3 are Chemistry, 2 are History books and 1 is a Language book.

i. How many different arrangements are possible? (No restriction.)

$$10!$$

ii. How many different arrangements are possible if the books of the same subject are to stay together?

$$\underset{\text{4 Math}}{\overset{4!}{\uparrow}} \quad \cdot \quad \underset{\text{3 Chem.}}{\overset{3!}{\uparrow}} \quad \cdot \quad \underset{\text{2 Hist.}}{\overset{2!}{\uparrow}} \quad \cdot \quad \underset{\text{1 Lang.}}{\overset{1!}{\uparrow}}$$

Now order these 4 groups in 4! different ways. So the total number is

$$4! \cdot 4! \cdot 3! \cdot 2! \cdot 1!$$

iii. What if we only want Math books to stay together?

We can consider all Math books as a single group, and treat the collection as a collection of 7 books in total. The number of permutations of 7 books is 7!. Moreover, the Math books permute among themselves in 4!. Hence the total number of permutations is

$$4! \cdot 7!$$

iii. What if we keep Math books together and Chem. books together?

We can consider all Math books as a single group, all Chem books as a single group, and treat the collection as a collection of 5 books in total. The number of permutations of 5 books is 5!. Moreover, the Math books permute among themselves in 4!, the Chem books permute among themselves in 3!. Hence the total number of permutations is

$$4! \cdot 3! \cdot 5!$$

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■ **Example 1.4** How many different 3 letter words can you write using the letters of PEP? Here we note that there are 2 P's. If we index these two P's as P_1 and P_2 then our way leads to the following words:

$$\left.\begin{array}{l} P_1EP_2 \\ P_2EP_1 \\ P_1P_2E \\ P_2P_1E \\ EP_1P_2 \\ EP_2P_1 \end{array}\right\} \rightarrow 6$$

However, we know that the words of the same color above are the same. So they should be counted only once instead of twice. That's why, we should ignore the letters originated due to the permutations of P's among themselves. To do this, we divide the total number by the number of repetitions of the same letter, which is 2! in this case. Hence we get

$$\frac{3!}{2!} = 3$$

different words. ■

In general, there are

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, ..., n_r are alike. Here we have

$$n = n_1 + n_2 + \ldots + n_r.$$

■ **Example 1.5** How many different words can you write using the letters of MISSISSIPPI? Clearly, we start with counting the number of permutations of 11 letters. But among these, we have 1 M, 4 S's, 2 P's and 4 I's. To eleminate the duplicates, we divide 11! by the number of permutations of S's among themselves, than by the number of permutations of P's among themselves, and than by the number of permutations of I's among themselves.

$$\frac{11!}{4! \cdot 2! \cdot 4! \cdot 1!}.$$

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■ **Example 1.6** A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the U.S., 2 are German and 1 from Brazil. If the tournament's result lists just the nationalities of players in order in which they placed, how many orderings are possible?
A sample list will be:

$$\begin{array}{l} 1. Germany \\ 2. U.S. \\ 3. Russia \\ 4. Brazil \\ 5. Germany \\ 6. Russia \\ 7. U.S. \\ 8. U.S. \\ 9. Russia \\ 10. Russia \end{array}$$

There are objects of the same type. Hence the number of orderings is

$$\frac{10!}{4! \cdot 3! \cdot 2! \cdot 1!}.$$

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