Littlewood-Paley theory, which is the main theme of my recent research, was developed to study function spaces in harmonic analysis and partial differential equations. Given a function $f$ in $L^p(\mathbb{R}^d)$, one defines another function that depends on $f$ in a nonlinear way and whose $L^p$ norm is comparable to the $L^p$ norm of $f$. Littlewood-Paley theory is useful in determining which linear operators are bounded in $L^p$, and these estimates are used in PDE to prove existence results for wide classes of equations.

On the analytical side, Littlewood and Paley discussed these functions in 1930’s in the one-dimensional setting. They used the theory to investigate of convergence of Fourier series using complex analysis techniques. Later, in the 1970’s, E. M. Stein pioneered the use of real variable methods involving Littlewood-Paley functions in the 1970’s, and his work allowed higher-dimensional extensions of the theory.

Littlewood-Paley theory is a powerful tool to prove the boundedness of certain operators that arise via singular integrals. The prototype of such operators is the Hilbert transform. Operators similar to the Hilbert transform can be characterized by their corresponding Fourier multipliers, and in many cases, it is easier to work with multipliers rather than to work with the operators themselves. Mihlin and Marcinkiewicz each proved a multiplier theorem. Mihlin’s theorem says that if $m$ is a Fourier multiplier whose partial derivatives of order $\gamma$ can be bounded above by a constant multiple of $|x|^{-|\gamma|}$ for appropriate $\gamma$, then the corresponding operator is a bounded operator on $L^p$. In the one-dimensional case, there is a neat probabilistic proof based on the theory of Brownian motion.

Independently of Mihlin, Marcinkiewicz proved another multiplier theorem in 1939. He proved that for a bounded function on $\mathbb{R}^d$ with continuous partial derivatives of order $\gamma$ for suitable $\gamma$ to be an $L^p$ multiplier, it is enough to control the integral

$$\int_{I_1 \times \ldots \times I_d} |\partial^{\gamma} m(x)| \, dx$$

over all dyadic boxes $I_1 \times \ldots \times I_d$. The analytic approach to these multiplier theorems can be found in [2].

Besides the analytic methods, a probabilistic approach has been used since the 1970’s. P.A.Meyer and E.M.Stein considered the case where a function has a harmonic exten-
sion provided that it is continuous at the boundary and the domain satisfies certain conditions. The method used in their work was based on Brownian motion. By combining the boundary function with a continuous stochastic process, Brownian motion, one can obtain the harmonic extension to the whole domain. Considering the upper-half space $\mathbb{R}^d \times \mathbb{R}^+$, and an $L^p$-function, say $f$, on the boundary $\mathbb{R}^d \times \{0\}$, the Littlewood-Paley function is defined as

$$G_f(x) = \left( \int_0^\infty t \cdot |\nabla u(x, t)|^2 \, dt \right)^{1/2}$$

where $u$ is the harmonic extension of $f$. The comparability of the norms of $f$ and $G_f$ can be proved using the theory of Brownian motion. This and many other results of Littlewood-Paley theory were studied thoroughly in the continuous case. (The details can be found in [1].)

In the last 20 years, there has been an increasing interest in non-continuous stochastic processes. In particular symmetric $\alpha$-stable processes play an important role in today’s probability theory, and there has been a remarkable increase in the number of applications of symmetric $\alpha$-stable processes.

My main goal was to study Littlewood-Paley functions where Brownian motion is replaced by a symmetric $\alpha$-stable process. Symmetric $\alpha$-stable processes and their connections with harmonic analysis has been studied by N. Bouleau, D. Lamberton, N. Varopoulos and P.A. Meyer in 1980’s. However, Littlewood-Paley functions in the discontinuous case have remained an open question until today.

On the half space, $\mathbb{R}^d \times \mathbb{R}^+$, I focus on a product process $X_t = (Y_t, Z_t)$, where $Y_t$ is a $d$-dimensional symmetric $\alpha$-stable process and $Z_t$ is a 1-dimensional Brownian motion. If $f$ is an $L^p$ function on $\mathbb{R}^d$, then it is known that the harmonic extension of $f$ to $\mathbb{R}^d \times \mathbb{R}^+$ is given by $u(x, t) = \mathbb{E}^{(x,t)}(f(Y_{T_0}))$, where $T_0$ denotes the first exit time of the product process $X_t$ from the half-space $\mathbb{R}^d \times \mathbb{R}^+$, and $\mathbb{E}^{(x,t)}(\cdot)$ denotes the expectation with respect to the probability measure $\mathbb{P}^{(x,t)}$. In this case, my studies has shown that the Littlewood-Paley function $G_f$ should be written as a combination of continuous and discontinuous components:
Converting the results obtained in the continuous case, one can ask if the analogous ones hold in this new setting. Comparability of the $L^p$-norms of the functions $f$ and $G_f$ is the fundamental property which one needs in order to study this theory further. I accomplished the relations $\|G_f\|_p \leq c\|f\|_p$ for $p \in (2, \infty)$, $\|G_f\|_2 = c\|f\|_2$, and $\|f\|_p \leq c\|G_f\|_p$. Here I used the composition of these functions with the process, and hence I was able to use the power of martingale theory. Now I am in the process of completing the last equality $\|G_f\|_p \leq c\|f\|_p$ for $p \in (1, 2)$. This last inequality requires strong mathematical tools such as a Harnack inequality and a Green’s identity, which were not known in this setup. I proved a version of the Harnack inequality, which states that there is a constant $c$ such that for any function $u$ that is harmonic with respect to the product process, we have $u(x) \leq cu(y)$ for any $x$ and $y$ in a suitable rectangular box.

The last part of my research focuses on the multiplier problem. Using the Littlewood-Paley theory that I developed, I have been able to obtain an interesting multiplier theorem.

FUTURE RESEARCH

In the continuous case, an important use of Littlewood-Paley functions is to prove the multiplier theorems of Mihlin and Marcinkiewicz. If $T$ is a convolution operator, $Tf = K \ast f$, so that $\hat{(Tf)} = m \hat{f}$, then we can give sufficient conditions on $m$ so that $T$ will map $L^p$ to $L^p$. As I mentioned above, the Mihlin and Marcinkiewicz multiplier theorems provide two different sets of necessary conditions for such operators. These multipliers are obtained from the Littlewood-Paley function $G_f(x) = \left(\int_0^\infty t \cdot |\nabla u(x, t)|^2 dt\right)^{1/2}$. In our setup, we obtained a different Littlewood-Paley function. So it is natural to expect to have different multipliers. In the near future, I want to investigate these new multipliers and see if such a characterization as Mihlin’s or
Marcinkiewicz’s can be given when Brownian motion is replaced by the discontinuous process $X_t$. This will perhaps allow us to study some new types of Fourier multipliers, and perhaps obtain new results not yet known to analysts.

Another interesting question is about the space BMO, the space of bounded mean oscillation, and the spaces $H^p$, the Hardy spaces. For example, in the one dimensional case, the Hilbert transform is not bounded on $L^1$. However, it is bounded on $H^1$. Do the Littlewood-Paley functions I have been studying extend as bounded operators when we replace $L^1$ by $H^1$ and the dual space of $L^1$, which is $L^\infty$, with the dual of $H^1$, which is BMO. If so, we could then can discuss properties concerning the $H^p$ boundedness of singular integrals.

I also intend to see if my results can be extended to Banach space-valued functions. In the continuous case, Banach space-valued martingales have been studied by several mathematicians, including Hytönen and McConnell. McConnell presented extensions of the Mihlin multiplier theorem and Littlewood-Paley inequalities to the case of Banach valued functions in 1984. I want to see if I can obtain such an extension for the new Littlewood-Paley functions which originate from the process $X_t$. Later, the techniques used for the extension of Littlewood-Paley inequalities might lead us an extension of the Mihlin multiplier theorem as well.

References
