GENERALIZED PROCRUSTES ANALYSIS AND ITS APPLICATIONS IN PHOTOGRAMMETRY

M. Devrim AKCA

Praktikum in Photogrammetrie, Fernerkundung und GIS

Table of Contents

Introduction

Procrustes analysis is a **least-squares solution** method of the **similarity transformation parameters** among **two or more** model point matrices, satisfying their maximal agreement.

- Algorithmically, there is no limit for the dimension **k** of the model point coordinates (In Geodetic Sciences usually **k** = 2,3).
- It has a linear functional model. No need to initial approximations for unknowns.
- It does not define and solve the classical normal equations system.

Who is Proctustes?

The name of the method comes from Greek Mythology.

Procrustes, or "one who stretches" was a robber in Greek Mythology. He preyed on his victims offered a magical bed that would fit any guest. He then either stretched the guests or cut off their limbs to make them fit perfectly into the bed.

The method was explained and named by P. Schoenemann who is a scientist in the Quantitative Psychology area.

(Schoenemann, 1966)
Orthogonal Procrustes $E = AT - B$

(Schoenemann and Carroll, 1970)
Extended Orthogonal Procrustes $E = cAT + jt^T - B$

Similar methods in Computer Vision and Robotics (Arun et al.1987, Horn et al.1988)

(Gower, 1975, Ten Berge, 1977)
Generalized Orthogonal Procrustes

(Lissitz et al., 1976, Koschat and Swayne, 1991, Goodall, 1991)
Weighted Procrustes

Extended Orthogonal Procrustes Analysis (EOP)

The problem is least squares fitting of a given matrix **A** to another given matrix **B:**

$$E = cAT + jt^T - B \qquad (1)$$

- $j^T = [1\ 1\ \dots\ 1]$ is unit vector (**1** x **p**)
- **A** and **B** are point matrices (**p** x **k**)
- **E** is random error matrix (**p** x **k**)
- **T** is unknown orthogonal rotation matrix (**k** x **k**)
- **t** is unknown translation vector (**k** x **1**)
- **c** is unknown scale factor (scalar)
- **p** is the number of common points
- **k** is the number of dimensions

In order to obtain the least squares estimation of the unknowns (**T**, **t**, **c**) let us write the Lagrangean function:

$$F = tr\{E^T E\} + tr\{L(T^T T - I)\} \qquad (2)$$

$$F = tr\{(cAT + jt^T - B)^T (cAT + jt^T - B)\} + tr\{L(T^T T - I)\} \qquad (3)$$

The derivations of the Lagrangean function with respect to unknowns must be set to zero in order to satisfy [vv]=min condition:

$$\frac{\partial \mathbf{F}}{\partial \mathbf{T}} = 2c^2 \underbrace{A^T A}_{\text{Symm.}} T - 2cA^T B + 2cA^T j t^T + T\left(\underbrace{L + L^T}_{\text{Symm.}}\right) = \mathbf{0} \qquad (4)$$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{t}} = 2p\,t - 2B^T j + 2cT^T A^T j = \mathbf{0} \quad , \quad p = j^T j \qquad (5)$$

Left multiply by T^T

$$\frac{\partial \mathbf{F}}{\partial \mathbf{c}} = 2c\,\mathrm{tr}\left\{T^T A^T A T\right\} - 2\,\mathrm{tr}\left\{B^T A T\right\} + 2\,\mathrm{tr}\left\{T^T A^T j t^T\right\} = \mathbf{0} \qquad (6)$$

$$c^2 T^T\left(A^T A\right)T - cT^T A^T B + cT^T A^T j t^T + \frac{L + L^T}{2} = 0 \qquad (7)$$

$$\underbrace{\frac{\left(L + L^T\right)}{2}}_{\text{Symm.}} = \underbrace{cT^T A^T B - cT^T A^T j t^T}_{\text{Must be symm.}} - c^2 T^T \underbrace{\left(A^T A\right)}_{\text{Symm.}} T = \underbrace{\left[\frac{\left(L + L^T\right)}{2}\right]^T}_{\text{Symm.}} \qquad (8)$$

In equation (5):

$$\frac{\partial \mathbf{F}}{\partial \mathbf{t}} = 2p\,\mathbf{t} - 2B^T j + 2cT^T A^T j = \mathbf{0} \quad \Longrightarrow \quad \mathbf{t} = \left(B - cAT\right)^T j/p \qquad (9)$$

$$\underbrace{T^T A^T B - T^T A^T j t^T = \text{symm.}}_{\text{substitution}} \qquad (10)$$

$$\underbrace{T^T A^T B - T^T A^T \left(\frac{j j^T}{p}\right) B}_{\text{Must be symmetric}} + \underbrace{c T^T A^T \left(\frac{j j^T}{p}\right) A T}_{\text{Symm.}} = \text{symm.} \qquad (11)$$

$$T^T \underbrace{A^T \left(I - \frac{j j^T}{p}\right) B}_{\text{Let say } \mathbf{S}} = \text{symm.} \quad \Longrightarrow \quad T^T S = \text{symm} \quad \Longrightarrow \quad \boxed{T^T S = S^T T}$$

($\mathbf{k}$ x $\mathbf{k}$) dimensional

Left multiply by T

$$T^T S = S^T T$$

Right multiply by T^T

$$S = T S^T T \qquad T^T S T^T = S^T$$

$$\underbrace{S S^T}_{\text{Symm.}} = T \underbrace{S^T S}_{\text{Symm.}} T^T \tag{12}$$

$$\mathrm{svd}\{SS^T\} = T\ \mathrm{svd}\{S^T S\}\ T^T$$

svd{ }: Singular Value Decomposition

$$\mathbf{V D_S V^T} = \mathbf{T W D_S W^T T^T}$$

$\mathbf{D_S}$: diagonal eigenvalue matrix

$\mathbf{V},\mathbf{W}$: orthonormal eigenvector matrices

$$\Rightarrow V = TW \Rightarrow \boxed{\mathbf{T} = \mathbf{V W^T}} \tag{13}$$

$$\mathrm{svd}\{S\} = \mathrm{svd}\left\{A^T\left(I - \frac{jj^T}{p}\right)B\right\} = VDW^T \quad , \quad D \neq D_s$$

$$\mathbf{T} = \mathbf{V}\mathbf{W}^{\mathrm{T}} \quad (13)$$

$$\mathbf{t} = \left(\mathrm{B} - \mathrm{cAT}\right)^{\mathrm{T}} \mathrm{j}/\mathrm{p}$$

Equation (9)

substitution

$$\frac{\partial \mathbf{F}}{\partial \mathbf{c}} = 2\mathrm{c}\,\mathrm{tr}\left\{\mathrm{T}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{AT}\right\} - 2\,\mathrm{tr}\left\{\mathrm{B}^{\mathrm{T}}\mathrm{AT}\right\} + 2\,\mathrm{tr}\left\{\mathrm{T}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{j}\,\mathrm{t}^{\mathrm{T}}\right\} = \mathbf{0}$$

$$\mathbf{c} = \frac{\mathrm{tr}\left\{\mathbf{T}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\left(\mathbf{I} - \frac{\mathbf{j}\mathbf{j}^{\mathrm{T}}}{\mathbf{p}}\right)\mathbf{B}\right\}}{\mathrm{tr}\left\{\mathbf{A}^{\mathrm{T}}\left(\mathbf{I} - \frac{\mathbf{j}\mathbf{j}^{\mathrm{T}}}{\mathbf{p}}\right)\mathbf{A}\right\}} \quad (14)$$

Finally, translation vector can be solved from Equation (9)

$$\mathbf{t} = \left(\mathrm{B} - \mathrm{cAT}\right)^{\mathrm{T}} \mathrm{j}/\mathrm{p} \quad (15)$$

Weighted Extended Orthogonal Procrustes Analysis (WEOP)

WEOP can directly calculate the least-squares estimation of the similarity transformation parameters between two model point matrices, in which points are differently weighted.

$$\mathrm{tr}\left\{\left(c\mathbf{A}\mathbf{T}+\mathbf{j}\mathbf{t}^{\mathrm{T}}-\mathbf{B}\right)^{\mathrm{T}}\underset{p\ \mathrm{x}\ p}{\mathbf{W}_{\mathrm{P}}}\left(c\mathbf{A}\mathbf{T}+\mathbf{j}\mathbf{t}^{\mathrm{T}}-\mathbf{B}\right)\underset{k\ \mathrm{x}\ k}{\mathbf{W}_{\mathrm{K}}}\right\}=\min \qquad \text{LS cond.}$$

$$\mathbf{T}^{\mathrm{T}}\mathbf{T}=\mathbf{T}\,\mathbf{T}^{\mathrm{T}}=\mathbf{I} \qquad \text{Orthogonality cond.}$$

- Let us assume that $\mathbf{W}_{\mathbf{K}}=\mathbf{I}$ [*If* $\mathbf{W}_{\mathrm{K}}\neq\mathbf{I}$ *, solution is iterative (Koschat et. 1991)*]
- Let us treat to obtain a similar expression as **EOP**

$$\mathrm{W}_{\mathrm{P}}=\mathrm{Q}^{\mathrm{T}}\mathrm{Q} \qquad \text{(Cholesky decomposition)} \qquad (16)$$

$$\mathrm{tr}\left\{\left(c\mathrm{A}\mathrm{T}+\mathrm{j}\mathrm{t}^{\mathrm{T}}-\mathrm{B}\right)^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\left(c\mathrm{A}\mathrm{T}+\mathrm{j}\mathrm{t}^{\mathrm{T}}-\mathrm{B}\right)\mathbf{I}\right\}=\min$$

$$\mathrm{tr}\left\{\left(c\mathbf{Q}\mathrm{A}\mathrm{T}+\mathbf{Q}\,\mathbf{j}\,\mathrm{t}^{\mathrm{T}}-\mathbf{Q}\mathbf{B}\right)^{\mathrm{T}}\left(c\mathbf{Q}\mathrm{A}\mathrm{T}+\mathbf{Q}\,\mathbf{j}\,\mathrm{t}^{\mathrm{T}}-\mathbf{Q}\mathbf{B}\right)\right\}=\min \qquad (17)$$

By substituting $\mathbf{A}_w = \mathbf{Q.A}$, $\mathbf{B}_w = \mathbf{Q.B}$, and $\mathbf{j}_w = \mathbf{Q.j}$

$$\mathrm{tr}\left\{ \left(c\mathbf{A}_w \mathrm{T} + \mathbf{j}_w \mathrm{t}^T - \mathbf{B}_w \right)^T \left(c\mathbf{A}_w \mathrm{T} + \mathbf{j}_w \mathrm{t}^T - \mathbf{B}_w \right) \right\} = \min \qquad (18)$$

This is the same expression as Extended Orthogonal Procrustes (EOP) analysis. Therefore this problem can be solved by the same formulas:

$$\mathrm{svd}\left\{ \mathrm{A}_w^T \left(\mathrm{I} - \frac{\mathrm{j}_w \mathrm{j}_w^T}{\mathrm{j}_w^T \mathrm{j}_w} \right) \mathrm{B}_w^T \right\}_{k\,x\,k} = \mathrm{VDW}^T$$

$$\mathbf{T} = \mathbf{V}\,\mathbf{W}^T \qquad (19)$$

$$\mathbf{c} = \mathrm{tr}\left\{ \mathbf{T}^T \mathbf{A}_w^T \left(\mathbf{I} - \frac{\mathbf{j}_w \mathbf{j}_w^T}{\mathbf{j}_w^T \mathbf{j}_w} \right) \mathbf{B}_w \right\} \Big/ \mathrm{tr}\left\{ \mathbf{A}_w^T \left(\mathbf{I} - \frac{\mathbf{j}_w \mathbf{j}_w^T}{\mathbf{j}_w^T \mathbf{j}_w} \right) \mathbf{A}_w \right\} \qquad (20)$$

$$\mathbf{t} = \left(\mathbf{B}_w - \mathbf{c}\mathbf{A}_w \mathbf{T} \right)^T \frac{\mathbf{j}_w}{\mathbf{j}_w^T \mathbf{j}_w} \qquad (21)$$

Generalized Orthogonal Procrustes Analysis (GP)

GP provides the least-squares correspondence of **m** (m>2) model points matrices. It satisfies the following least squares objective function

$$\mathrm{tr}\left\{\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left[\left(c_i\mathbf{A}_i\mathbf{T}_i+\mathbf{j}\mathbf{t}_i^{T}\right)-\left(c_j\mathbf{A}_j\mathbf{T}_j+\mathbf{j}\mathbf{t}_j^{T}\right)\right]^{T}\left[\left(c_i\mathbf{A}_i\mathbf{T}_i+\mathbf{j}\mathbf{t}_i^{T}\right)-\left(c_j\mathbf{A}_j\mathbf{T}_j+\mathbf{j}\mathbf{t}_j^{T}\right)\right]\right\}=\min$$

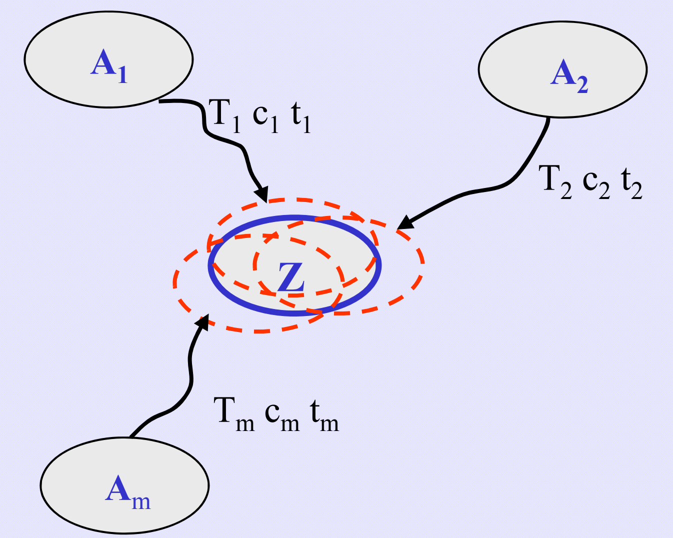
The solution of the problem is searching of the unknown **optimal** matrix **Z** (also named **consensus matrix**).

$$\mathbf{Z}+\mathbf{E}_i=\hat{\mathbf{A}}_i=c_i\mathbf{A}_i\mathbf{T}_i+\mathbf{j}\mathbf{t}_i^{T}\qquad,\quad i=\{1,2,\ldots,m\}$$

$$\mathrm{vec}(\mathbf{E}_i)\sim N\left\{0,\boldsymbol{\Sigma}=\sigma^2\left(\mathbf{Q}_P\otimes\mathbf{Q}_K\right)\right\}$$

Kronocker product

Covariance matrix



GP concept (Crosilla&Beinat, 2002)

In the literature, there are many solution methods. Only one of them will be explained here.

$$\mathbf{C} = \frac{1}{m}\sum_{i=1}^{m}\hat{\mathbf{A}}_i$$

Geometrical centroid of the transformed matrices

The centroid **C** corresponds the least squares estimation of the true value **Z** (Crosilla and Beinat 2002).

$$\sum_{i<j}^{m}\left\| \hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j \right\|^2 = \sum_{i<j}^{m} \mathrm{tr}\left\{ \left(\hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j\right)^T \left(\hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j\right) \right\}$$

$$m\sum_{i=1}^{m}\left\| \hat{\mathbf{A}}_i - \mathbf{C} \right\|^2 = m\sum_{i=1}^{m} \mathrm{tr}\left\{ \left(\hat{\mathbf{A}}_i - \mathbf{C}\right)^T \left(\hat{\mathbf{A}}_i - \mathbf{C}\right) \right\}$$

The above two objective functions are equivalent (Kristof and Wingersky, 1971, Borg and Groenen, 1997).

Initialize:
Define the initial centroid **C**

Iterate:
(1) Direct solution of similarity transformation parameters of each $\mathbf{A}_i$ with respect to the centroid **C** by means of **WEOP** solution

(2) After the calculation of each matrix is carried out, iterative updating of the centroid **C**

Until: Global convergence, i.e. stabilization of the centroid **C**

Algorithmically, similar to **Separate Adjustment** (Wang, Clarke 2001).

Case 1: Different weights among the models

$$\mathrm{vec}(\mathbf{E}_i) \sim N\{0, \mathbf{\Sigma}_i = \sigma^2(\mathbf{Q}_{Pi} \otimes \mathbf{Q}_{Ki})\} \quad , \quad \mathbf{Q}_{Ki} = \mathbf{I} \quad , \quad \mathbf{Q}_{Pi} \neq \mathbf{I} \quad \text{(diagonal)}$$

Each row of $\hat{\mathbf{A}}_i$ has different dispersion with respect to the true value $\mathbf{Z}$ and the dispersion varies for each model points matrix **i**=1,2,…,**m**.

In this case, least-squares objective function and centroid $\mathbf{C}$ are as follows:

$$\sum_{i=1}^{m} \mathrm{tr}\left\{ (\hat{\mathbf{A}}_i - \mathbf{C})^T \mathbf{P}_i (\hat{\mathbf{A}}_i - \mathbf{C}) \right\} = \min \tag{22}$$

$$\mathbf{C} = \left(\sum_{i=1}^{m} \mathbf{P}_i \right)^{-1} \left(\sum_{i=1}^{m} \mathbf{P}_i \hat{\mathbf{A}}_i \right) \quad , \quad \mathbf{P}_i = \mathbf{Q}_{Pi}^{-1} \tag{23}$$

Case 2: Missing points/different weights among the models

In real applications, all of the **p** points could not be visible in all of the model points matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$. A diagonal binary (**p** x **p**) matrix $\mathbf{M}_i$ can be associated to every matrix $\mathbf{A}_i$, in which the diagonal elements are 1 or 0 , according to existence or absence of the point in the **i**-th model (Commandeur (1991).

$$\mathbf{A}_1 = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ - & - & - \\ - & - & - \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} - & - & - \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ - & - & - \\ x_6 & y_6 & z_6 \end{bmatrix} \qquad \mathbf{A}_3 = \begin{bmatrix} - & - & - \\ - & - & - \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{bmatrix}$$

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \qquad \mathbf{M}_3 = \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

In the case of combined **weighted/missing** point solution, least-squares objective function and centroid $\mathbf{C}$ are as follows:

$$\mathbf{D}_i = \mathbf{M}_i \mathbf{P}_i = \mathbf{P}_i \mathbf{M}_i \quad , \quad \mathbf{P}_i = \mathbf{Q}_{Pi}^{-1} \tag{24}$$

(diagonal)

$$\sum_{i=1}^{m} \mathrm{tr}\left\{ \left(\hat{\mathbf{A}}_i - \mathbf{C} \right)^T \mathbf{D}_i \left(\hat{\mathbf{A}}_i - \mathbf{C} \right) \right\} = \min \tag{25}$$

$$\boxed{\mathbf{C} = \left(\sum_{i=1}^{m} \mathbf{D}_i \right)^{-1} \left(\sum_{i=1}^{m} \mathbf{D}_i \hat{\mathbf{A}}_i \right)} \tag{26}$$

Case 3: Missing points/different weights among the models, and different weights among the coordinate components (Beinat, Crosilla, 2002)

$$\text{vec}(\mathbf{E}_i) \sim \text{N}\{0, \mathbf{\Sigma}_i = \sigma^2(\mathbf{Q}_{Pi} \otimes \mathbf{Q}_{Ki})\} \quad , \quad \mathbf{Q}_{Pi} \neq \mathbf{I} \quad \text{and} \quad \mathbf{Q}_{Ki} \neq \mathbf{I} \tag{27}$$

(diagonal)

In this case, least-squares objective function and centroid $\mathbf{C}$ are as follows:

$$\mathbf{D}_i = \mathbf{M}_i\mathbf{P}_i = \mathbf{P}_i\mathbf{M}_i \qquad \mathbf{P}_i = \mathbf{Q}_{Pi}^{-1} \qquad \mathbf{K}_i = \mathbf{Q}_{Ki}^{-1} \tag{28}$$

$$\sum_{i=1}^{m} \text{tr}\left\{(\hat{\mathbf{A}}_i - \mathbf{C})^T \mathbf{D}_i (\hat{\mathbf{A}}_i - \mathbf{C}) \mathbf{K}_i\right\} = \min \tag{29}$$

$$\text{vec}(\mathbf{C}) = \left(\sum_{i=1}^{m} \mathbf{K}_i \otimes \mathbf{D}_i\right)^{-1} \left[\sum_{i}^{m} \underbrace{\mathbf{K}_i \otimes \mathbf{D}_i}_{(\mathbf{kp\ x\ kp})} \underbrace{\text{vec}(\hat{\mathbf{A}}_i)}_{(\mathbf{kp\ x\ 1})}\right] \tag{30}$$

Applications in Photogrammetry

• Registration of laser scanner point clouds (Beinat, Crosilla, 2001)

• An adaptation of GP method into block adjustment by independent models (Crosilla, Beinat, 2002).

GP is a free solution, since the consensus matrix **Z** is in any orientation-position-scale in the k-dimensional space. Controversially, **conventional block adjustment by independent models** solution needs the datum definition.

Example 1: synthetic data

$$-\mathbf{e} = c\mathbf{A}\mathbf{T} + \mathbf{j}\mathbf{t}^{T} - \mathbf{B} \qquad \mathbf{e} \sim N\{\mu = 0,\ \sigma = \pm 5\text{mm}\}$$

p = 100 points
k = 3 dimension

	Iterations	Computation time (sec.)
Least-squares adjustment	3	0.09
WEOP	direct	0.03

*Numerically, **same** results for the unknown transformation parameters.*

Solution strategy for least-squares similarity transformation:

- initial approximations for unknowns: closed-form solution (Dewitt, 1996)
- classic solution: normal matrix partitioning, Cholesky decomposition, and back-substitution
- after the iterations, Q_{XX} calculation for theoretical precision
- Control points are treated as stochastic quantities

Computational Cores:

- **WEOP:** Singular Value Decomposition of **(k x k)** matrix
- **Least-squares adjustment:** well-known solution of **[(7+pk) x (7+pk)]** normal eq. matrix

Example 2: real data (laser scanner)

$$-\mathbf{e}_i = c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T - \mathbf{Z} \quad , \quad i = \{1,2,\ldots,5\} \qquad \mathrm{vec}(\mathbf{e}_i) \sim N\{0, \mathbf{\Sigma} = \sigma^2 I\} \qquad \sigma = \pm 3_{mm}$$

	Iterations	Computation times (sec.)	σ0 (mm.)
Block adjustment by independent models *	3	0.01	3.4
Generalized Orthogonal Procrustes (GP) **	6	0.01	2.2

m = 5 models
p = 10 points
k = 3 dimension

* *datum defined by 3 of the points || the comp.time also includes Q_{XX} calculation*
** *free solution || Sigma naught is with respect to centroid C*

For block adjustment by independent models method:

For $\mathbf{N}_{11}$:	**m** (**u** x **u**)	= 5 . (7 x 7)	= 245 variables
For $\mathbf{N}_{12}$:	(**m u**) x (**p k**)	= (5 .7) x (10 . 3)	= 1050 variables
For $\mathbf{N}_{22}$:	**p k**	= 10 . 3	= 30 variables
		Totally	= **10 600 Bytes**

For Generalized Orthogonal Procrustes (GP) method:

For unknowns of each model :	**m u**	= 5 . 7	= 35 variables
For centroid **C** :	**p k**	= 10 . 3	= 30 variables
		Totally	= **520 Bytes**

Example 3: synthetic data

$$-\mathbf{e}_i = c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T - \mathbf{Z} \quad , \quad i = \{1,2,\ldots,9\} \qquad e \sim N\{\mu = 0,\ \sigma = \pm 0.002_{\text{ unitless}}\}$$

	Iterations	Computation times (sec.)	σX (unitless)	σY (unitless)	σZ (unitless)
Block adjustment by independent models *	5	1.032	0.0018	0.0018	0.0019
Generalized Orthogonal Procrustes (GP) **	35	1.953	0.0017	0.0020	0.0019

m = 9 models
p = 100 points
k = 3 dimension

* ***30 control points as stochastic quantities || the comp.time also includes Q_{XX} calculation***
** ***30 control points (adaptation to block adjustment by independent models Crosilla, Beinat, 2002)***
|| Sigma naughts are with respect to centroid C

In the case of datum-definition, very slow convergence behavior of the Generalized Orthogonal Procrustes (GP) method compared to conventional block adjustment solution can be shown.

Comparison of GP method with the Conventional LS Solution

	Generalized Procrustes	**Conventional LS**
Linearity	Direct solution	Non-linear, needs to initial approx. Closed-form sol.
Limit for number of **k** dimensions	**No limit , flexible**	For **k > 3** , needs re-arrangement of the model
Datum definition	Free solution	Can be achievable by means of inner constraints
Stochastic model	Weak	**Powerful**
Computational core	SVD of (**k x k**) matrix	Solution of (**u x u**) normal matrix
Convergence	Slow	**Quick**
Speed	Almost equal	
Memory requirement	**Drastically less than**	More than
Theoretical Precision indicators	Weak	**Powerful**
Reliability indicators	Not available	**Powerful**

Conclusions

The most important disadvantage of the Procrustes method is lack of reliability criterion in order to detect and localize the blunders, which might be included by the data set. Without such a tool, the results that produced by the Procrustes method can be wrong in the case of existence of blunders in the data set.

THANK YOU!