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GENERALIZED PROCRUSTES ANALYSIS AND ITS APPLICATIONS IN PHOTOGRAMMETRY

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1. INTRODUCTION

Some measurement systems and methods can produce directly 3D coordinates of the relevant object with respect to a local coordinate system. Depending on the extension and shape complexity of the object, it may require two or more viewpoints in order to cover the object completely. These different local coordinate systems must be combined into a common system. This geometric transformation process is known as registration. The fundamental problem of the registration process is estimation of the transformation parameters.

In the context of traditional least-squares adjustment, the linearisation and initial approximations of the unknowns in the case of 3 or more dimensional similarity transformations are needed due to non-linearity of the functional model.

Procrustes analysis theory is a set of mathematical least-squares tools to directly estimate and perform simultaneous similarity transformations among the model point coordinates matrices up to their maximal agreement. It avoids the definition and solution of the classical normal equation systems. No prior information is requested for the geometrical relationship existing among the different model objects components. By this approach, the transformation parameters are computed in a direct and efficient way based on a selected set of corresponding point coordinates (Beinat and Crosilla, 2001).

The method was explained and named as **Orthogonal Procrustes** problem by Schoenemann (1966) who is a scientist in the Quantitative Psychology area. In this publication, Schoenemann gave the direct least-squares solution of the problem that is to transform a given matrix **A** into a given matrix **B** by an orthogonal transformation matrix **T** in such a way to minimize the sum of squares of the residual matrix $\mathbf{E} = \mathbf{AT} - \mathbf{B}$. The first generalization to the Schoenemann (1966) orthogonal Procrustes problem was given by Schoenemann and Carroll (1970) when a least squares method for fitting a given matrix **A** to another given matrix **B** under choice of an unknown rotation, an unknown translation and an unknown scale factor was presented. This method is often identified in statistics and psychometry as **Extended Orthogonal Procrustes** problem. After Schoenemann (1966), similar methods were proposed in computer vision and robotics area (Arun et al., 1987, and Horn et al., 1988).

The solution of the **Generalized Orthogonal Procrustes** problem to a set of more than two matrices was reported (Gower, 1975, Ten Berge, 1977). Further generalization in the stochastic model is called **Weighted Procrustes Analysis**, which can be different weighting across columns (Lissitz et al., 1976) or across rows (Koschat and Swayne, 1991) of a matrix configuration. An approach that can differently weight the homologous points coordinates was given (Goodall, 1991). A method that can take into account the stochastic properties of the coordinate axes was given by Beinat and Crosilla (2002).

Implementation details and two different applications of Procrustes Analysis in Geodetic Sciences were given by Crosilla and Beinat (2002, 2001): photogrammetric block adjustment by independent models, and registration of laser scanner point clouds. The reader can also find a detailed survey of the Procrustes analysis and its some possible applications in the Geodetic Sciences in (Crosilla, 1999).

The report is organized as follows. In the second section, the mathematical background and the algorithmic aspects of the Procrustes analysis is given. In the third section, two different applications of the Procrustes analysis in photogrammetry are presented. This section also compares the Procrustes Analysis and the conventional Least-Squares solution with respect to accuracy, computational cost, and operator handling. Discussion and conclusion are given in the fourth section.

2. PROCRUSTES ANALYSIS: THEORY AND ALGORITHMS

2.1. Who is Procrustes?

The name of the method comes from Greek Mythology (Figure 1). **Procrustes**, or "one who stretches," (also known as Prokrustes or Damastes) was a robber in the myth of Theseus . He preyed on travelers along the road to Athens. He offered his victims hospitality on a magical bed that would fit any guest. He then either stretched the guests or cut off their limbs to make them fit perfectly into the bed. Theseus, travelling to Athens to claim his inheritance, encountered the thief. The hero cut off the evil-doer's head to make him fit into the bed in which many "guests" had died (Greek Mythology Reference).

Figure 1: Procrustes in Greek Mythology (Procrustes Accommodaties ob maat)

2.2. Orthogonal Procrustes Analysis

Orthogonal Procrustes problem (Schoenemann, 1966) is the least squares solution of the problem that is the transformation of a given matrix **A** into a given matrix **B** by an orthogonal transformation matrix **T** in such a way to minimize the sum of squares of the residual matrix **E = AT - B**. Matrices **A** and **B** are (**p** x **k**) dimensional, in which contain **p** corresponding points in the **k-**dimensional space. A Least squares solution must satisfy the following condition

$$\mathrm{tr}\{E^T E\} = \mathrm{tr}\{(AT - B)^T (AT - B)\} = \min \qquad (1)$$

The problem also has another condition, which is the orthogonal transformation matrix,

$$T^T T = T T^T = I \qquad (2)$$

Both of the conditions can be combined in a Lagrangean function,

$$F = \mathrm{tr}\{E^T E\} + \mathrm{tr}\{L(T^T T - I)\} \qquad (3)$$

$$F = tr\{(AT - B)^T (AT - B)\} + tr\{L(T^T T - I)\} \quad (4)$$

$$F = tr\{T^T A^T AT - T^T A^T B - B^T AT + B^T B\} + tr\{L(T^T T - I)\} \quad (5)$$

where **L** is a matrix of Lagrangean multipliers, and **tr{ }** stands for trace of the matrix. The derivation of this function with respect to unknown **T** matrix must be set to zero.

$$\frac{\partial F}{\partial T} = 2A^T AT - 2A^T B + T(L + L^T) = 0 \quad (6)$$

where $(A^T A)$ and $(L+L^T)$ are symmetric matrices. Let us multiply equation (6) on the left side by $\mathbf{T^T}$,

$$T^T A^T AT - T^T A^T B + \frac{L + L^T}{2} = 0 \quad (7)$$

$$\frac{(L + L^T)}{2} = T^T (A^T B) - T^T (A^T A) T = \left[\frac{(L + L^T)}{2}\right]^T \quad (8)$$

Since $T^T(A^T A)T$ is symmetric, $T^T(A^T B)$ must also be symmetric. Remind that $(L+L^T)$ is also symmetric. Therefore, the following condition must be satisfied.

$$T^T (A^T B) = (A^T B)^T T \quad (9)$$

Multiplying Equation (9) on the left side by **T**,

$$(A^T B) = T(A^T B)^T T \quad (10)$$

and on the right side by $\mathbf{T^T}$

$$T^T (A^T B) T^T = (A^T B)^T \quad (11)$$

Finally, we have the following equation using Equations (10) and (11),

$$(A^T B)(A^T B)^T = T(A^T B)^T (A^T B) T^T \quad (12)$$

Matrices $[(A^T B)(A^T B)^T]$ and $[(A^T B)^T (A^T B)]$ are symmetric. Both of them have same eigenvalues.

$$svd\left\{(A^T B)(A^T B)^T\right\} = T \, svd\left\{(A^T B)^T (A^T B)\right\} T^T \quad (13)$$

where **svd{ }** stands for Singular Value Decomposition, namely Eckart-Young Decomposition. The result is,

$$V D_s V^T = T \, W D_s W^T T^T \quad (14)$$

This means that,

$$V = T\,W \tag{15}$$

Finally, we can solve the unknown orthogonal transformation matrix **T**.

$$\mathbf{T} = \mathbf{V}\,\mathbf{W}^{\mathbf{T}} \tag{16}$$

2.3. Extended Orthogonal Procrustes Analysis (EOP)

The first generalization to the Schoenemann (1966) orthogonal Procrustes problem was given by Schoenemann and Carroll (1970) when a least squares method for fitting a given matrix **A** to another given matrix **B** under choice of an unknown rotation **T**, an unknown translation **t** , and an unknown scale factor **c** was presented. This method is often identified in statistics and psychometry as **Extended Orthogonal Procrustes** problem. The functional model is the following

$$E = cAT + j\,t^{T} - B \tag{17}$$

where $\mathbf{j}^{T} = [1 \;\; 1 \;\; \ldots \;\; 1]$ is (**1** x **p**) unit vector, matrices **A** and **B** are (**p** x **k**) corresponding point matrices as mentioned before, **T** is (**k** x **k**) orthogonal rotation matrix, **t** is (**k** x **1**) translation vector, and **c** is scale factor. In order to obtain the least squares estimation of the unknowns (**T**, **t**, **c**) let us write the Lagrangean function

$$F = tr\{E^{T}E\} + tr\{L(T^{T}T - I)\} \tag{18}$$

$$F = tr\left\{(cAT + j\,t^{T} - B)^{T}(cAT + j\,t^{T} - B)\right\} \; + \; tr\{L(T^{T}T - I)\} \tag{19}$$

where

$$tr\{E^{T}E\} = tr\{B^{T}B\} + c^{2}tr\{T^{T}A^{T}AT\} + p\,t^{T}t - 2c\,tr\{B^{T}AT\} - 2\,tr\{B^{T}j\,t^{T}\} + 2c\,tr\{T^{T}A^{T}jt^{T}\}$$

and $\mathbf{p} = j^{T}j$ is a scalar, namely number of rows of the data matrices. The derivations of the Lagrangean function with respect to unknowns must be set to zero in order to obtain a least squares estimation,

$$\frac{\partial F}{\partial T} = 2c^{2}A^{T}AT - 2cA^{T}B + 2cA^{T}jt^{T} + T(L + L^{T}) = 0 \tag{20}$$

$$\frac{\partial F}{\partial t} = 2p\,t - 2B^{T}j + 2cT^{T}A^{T}j = 0 \tag{21}$$

$$\frac{\partial F}{\partial c} = 2c\,tr\{T^{T}A^{T}AT\} - 2\,tr\{B^{T}AT\} + 2\,tr\{T^{T}A^{T}j\,t^{T}\} = 0 \tag{22}$$

The translation vector from Equation (21) is that

$$t = (B - cAT)^T \, j/p \tag{23}$$

In Equation (20), (A^TA) and $(L+L^T)$ are symmetric matrices. Let us multiply Equation (20) on the left side by $\mathbf{T^T}$

$$c^2 T^T (A^T A) T - cT^T A^T B + cT^T A^T j t^T + \frac{L + L^T}{2} = 0 \tag{24}$$

$$\frac{(L + L^T)}{2} = cT^T A^T B - cT^T A^T jt^T - c^2 T^T (A^T A) T = \left[\frac{(L + L^T)}{2} \right]^T \tag{25}$$

Since $T^T(A^TA)T$ is symmetric, $[\ T^TA^TB - cT^TA^Tjt^T\ ]$ must also be symmetric. Remind that $(L+L^T)$ is also symmetric.

$$T^T A^T B - T^T A^T jt^T = \text{symm.} \tag{26}$$

According to Equation (23), Equation (26) can be written as

$$T^T A^T B - T^T A^T \left(\frac{jj^T}{p} \right) (B - cAT) = \text{symm.} \tag{27}$$

$$T^T A^T B - T^T A^T \left(\frac{jj^T}{p} \right) B + cT^T A^T \left(\frac{jj^T}{p} \right) AT = \text{symm.} \tag{28}$$

Since $T^T A^T \left(\frac{jj^T}{p} \right) AT$ is symmetric, the rest of the equation must also be symmetric,

$$T^T A^T B - T^T A^T \left(\frac{jj^T}{p} \right) B = \text{symm.} \tag{29}$$

$$T^T A^T \left[B - \left(\frac{jj^T}{p} \right) B \right] = \text{symm.} \tag{30}$$

$$T^T A^T \left(I - \frac{jj^T}{p} \right) B = \text{symm.} \tag{31}$$

Let us say,

$$S = A^T \left(I - \frac{jj^T}{p} \right) B \tag{32}$$

where matrix **S** is (**k** x **k**) dimensional. In order to satisfy Equation (31), the following condition must be satisfied. Note that transpose of the matrix is equal to itself, if the matrix is symmetric.

$$T^T S = S^T T \tag{33}$$

Multiplying Equation (33) on the left side by **T**,

$$S = TS^T T \tag{34}$$

and on the right side by $\mathbf{T}^\mathbf{T}$

$$T^T S T^T = S^T \tag{35}$$

Finally, we have the following equation using Equations (34) and (35),

$$SS^T = TS^T ST^T \tag{36}$$

Matrices $[SS^T]$ and $[S^T S]$ are symmetric. Both of them have same eigenvalues.

$$\text{svd}\{SS^T\} = T \cdot \text{svd}\{S^T S\} T^T \tag{37}$$

where **svd{ }** stands for Singular Value Decomposition, namely Eckart-Young Decomposition. The result is,

$$VD_s V^T = T\, WD_s W^T T^T \tag{38}$$

where matrices **V** and **W** are orthonormal eigenvector matrices, and $\mathbf{D_s}$ is the diagonal eigenvalue matrix. According to Equation (38),

$$V = TW \tag{39}$$

Finally, we can solve the unknown orthogonal transformation matrix **T**.

$$\mathbf{T} = \mathbf{V}\mathbf{W}^\mathbf{T} \tag{40}$$

In the calculation phase, one should take into account the following equation. According to (Schoenemann and Carroll, 1970)

$$\text{svd}\{\, S \,\} = \text{svd}\left\{ A^T \left(I - \frac{jj^T}{p} \right) B \right\} = VDW^T \quad , \quad D \neq D_s \tag{41}$$

In order to solve the scale factor **c**, let us substitute Equation (23) in Equation (22)

$$\mathbf{c} = \frac{\text{tr}\left\{ \mathbf{T}^T \mathbf{A}^T \left(\mathbf{I} - \frac{\mathbf{j}\mathbf{j}^T}{\mathbf{p}} \right) \mathbf{B} \right\}}{\text{tr}\left\{ \mathbf{A}^T \left(\mathbf{I} - \frac{\mathbf{j}\mathbf{j}^T}{\mathbf{p}} \right) \mathbf{A} \right\}} \tag{42}$$

Finally, translation vector **t** can be solved from Equation (23)

$$\mathbf{t} = (\mathbf{B} - \mathbf{c}\mathbf{A}\mathbf{T})^T\, \mathbf{j}/\mathbf{p} \tag{43}$$

2.4. Weighted Extended Orthogonal Procrustes Analysis (WEOP)

WEOP can directly calculate the least-squares estimation of the similarity transformation parameters between two differently weighted model point matrices. This aim is achieved when the following conditions are satisfied (Goodall, 1991).

$$\mathrm{tr}\left\{\left(c\mathbf{A}\mathbf{T}+\mathbf{j}\mathbf{t}^{\mathrm{T}}-\mathbf{B}\right)^{\mathrm{T}}\mathbf{W}_{\mathrm{P}}\left(c\mathbf{A}\mathbf{T}+\mathbf{j}\mathbf{t}^{\mathrm{T}}-\mathbf{B}\right)\mathbf{W}_{\mathrm{K}}\right\}=\min \tag{44}$$

$$\mathbf{T}^{\mathrm{T}}\mathbf{T}=\mathbf{T}\,\mathbf{T}^{\mathrm{T}}=\mathbf{I} \qquad \text{(orthogonalitiy condition)} \tag{45}$$

where matrices $\mathbf{A}$ and $\mathbf{B}$ are ($\mathbf{p}$ x $\mathbf{k}$) model point matrices, which contain the coordinates of $\mathbf{p}$ points in $R^{\mathbf{k}}$ space. Matrices $\mathbf{W_P}$ ($\mathbf{p}$ x $\mathbf{p}$) and $\mathbf{W_K}$ ($\mathbf{k}$ x $\mathbf{k}$) are optional weighting matrices of the $\mathbf{p}$ points and $\mathbf{k}$ components, respectively. Model points matrix $\mathbf{A}$ is transformed into best-fit of the model points matrix $\mathbf{B}$, by the unknown transformation parameters, namely orthogonal rotation matrix ($\mathbf{k}$ x $\mathbf{k}$) $\mathbf{T}$, translation vector ($\mathbf{k}$ x $\mathbf{1}$) $\mathbf{t}$, and scale factor $\mathbf{c}$. The vector $\mathbf{j}$ is ($\mathbf{p}$ x $\mathbf{1}$) unit vector.

At the first attempt, let us assume that $\mathbf{W}_{\mathrm{K}}=\mathbf{I}$, and let us re-arrange Equation (44) in order to obtain a similar expression as in Equation (19). For the sake of this aim, matrix $\mathbf{W_P}$ can be decomposed into lower and upper triangle matrices by Cholesky Decomposition.

$$W_{\mathrm{P}}=Q^{\mathrm{T}}Q \qquad \text{(Cholesky Decomposition)} \tag{46}$$

So that

$$\mathrm{tr}\left\{\left(cAT+jt^{\mathrm{T}}-B\right)^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\left(cAT+jt^{\mathrm{T}}-B\right)\mathbf{I}\right\}=\min \tag{47}$$

$$\mathrm{tr}\left\{\left(cT^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}+t\,\mathbf{j}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}-\mathbf{B}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\right)\left(c\mathbf{QA}T+\mathbf{Q}\,\mathbf{j}\,t^{\mathrm{T}}-\mathbf{QB}\right)\right\}=\min \tag{48}$$

Finally,

$$\mathrm{tr}\left\{\left(c\mathbf{QA}T+\mathbf{Q}\,\mathbf{j}\,t^{\mathrm{T}}-\mathbf{QB}\right)^{\mathrm{T}}\left(c\mathbf{QA}T+\mathbf{Q}\,\mathbf{j}\,t^{\mathrm{T}}-\mathbf{QB}\right)\right\}=\min \tag{49}$$

By substituting $\mathbf{A}_{\mathrm{w}}=\mathbf{Q.A}$, $\mathbf{B}_{\mathrm{w}}=\mathbf{Q.B}$, and $\mathbf{j}_{\mathrm{w}}=\mathbf{Q.j}$

$$\mathrm{tr}\left\{\left(c\mathbf{A}_{\mathrm{W}}T+\mathbf{j}_{\mathrm{W}}t^{\mathrm{T}}-\mathbf{B}_{\mathrm{W}}\right)^{\mathrm{T}}\left(c\mathbf{A}_{\mathrm{W}}T+\mathbf{j}_{\mathrm{W}}t^{\mathrm{T}}-\mathbf{B}_{\mathrm{W}}\right)\right\}=\min \tag{50}$$

Equation (50) is the same expression as in Equation (19). Therefore, this problem can be solved by the same method as in Extended Orthogonal Procrustes (EOP) analysis. Performing the Singular Value Decomposition of matrix product:

$$\mathrm{svd}\left\{A_{\mathrm{w}}^{\mathrm{T}}\left(I-\frac{j_{\mathrm{w}}j_{\mathrm{w}}^{\mathrm{T}}}{j_{\mathrm{w}}^{\mathrm{T}}j_{\mathrm{w}}}\right)B_{\mathrm{w}}^{\mathrm{T}}\right\}=VDW^{\mathrm{T}} \tag{51}$$

where **V** and **W** are orthonormal eigenvector matrices, and **D** is the diagonal eigenvalue matrix. Note that the dimension of **svd{ }** part is (**k** x **k**). The unknowns can be found as mentioned before.

$$\mathbf{T} = \mathbf{V}\,\mathbf{W}^{T} \tag{52}$$

$$\mathbf{c} = \mathrm{tr}\left\{\mathbf{T}^{T}\mathbf{A}_{w}^{T}\left(\mathbf{I} - \frac{\mathbf{j}_{w}\mathbf{j}_{w}^{T}}{\mathbf{j}_{w}^{T}\mathbf{j}_{w}}\right)\mathbf{B}_{w}\right\} \Big/ \mathrm{tr}\left\{\mathbf{A}_{w}^{T}\left(\mathbf{I} - \frac{\mathbf{j}_{w}\mathbf{j}_{w}^{T}}{\mathbf{j}_{w}^{T}\mathbf{j}_{w}}\right)\mathbf{A}_{w}\right\} \tag{53}$$

$$\mathbf{t} = \left(\mathbf{B}_{w} - \mathbf{c}\mathbf{A}_{w}\mathbf{T}\right)^{T} \frac{\mathbf{j}_{w}}{\mathbf{j}_{w}^{T}\mathbf{j}_{w}} \tag{54}$$

An iterative solution method for the case of $\mathbf{W}_{K} \neq \mathbf{I}$ was given by Koschat and Swayne (1991). Also, a direct solution method that can take into account the stochastic properties of the coordinate axes in the case of Generalized Orthogonal Procrustes Analysis (GP) was given by Beinat and Crosilla (2002). This method will be explained in the following section.

2.5. Generalized Orthogonal Procrustes Analysis (GP)

Generalized Procrustes Analysis is a well-known technique that provides least-squares correspondence of more than two model points matrices (Gower, 1975, Ten Berge, 1977, Goodall, 1991, Dryden and Mardia, 1998, Borg and Groenen, 1997). It satisfies the following least squares objective function:

$$\mathrm{tr}\left\{\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left[\left(c_{i}\mathbf{A}_{i}\mathbf{T}_{i} + \mathbf{j}\mathbf{t}_{i}^{T}\right) - \left(c_{j}\mathbf{A}_{j}\mathbf{T}_{j} + \mathbf{j}\mathbf{t}_{j}^{T}\right)\right]^{T}\left[\left(c_{i}\mathbf{A}_{i}\mathbf{T}_{i} + \mathbf{j}\mathbf{t}_{i}^{T}\right) - \left(c_{j}\mathbf{A}_{j}\mathbf{T}_{j} + \mathbf{j}\mathbf{t}_{j}^{T}\right)\right]\right\} = \min \tag{55}$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, … , $\mathbf{A}_m$ are **m** model points matrices, which contain the same set of **p** points in **k** dimensional **m** different coordinate systems. According to Goodall (1991), there is a matrix **Z** , also named **consensus matrix**, in which contains the **true** coordinates of the **p** points defined in a mean and common coordinate system (Figure 2). The solution of the problem can be thought as the search of the unknown **optimal** matrix **Z**.

$$\mathbf{Z} + \mathbf{E}_{i} = \hat{\mathbf{A}}_{i} = c_{i}\mathbf{A}_{i}\mathbf{T}_{i} + \mathbf{j}\mathbf{t}_{i}^{T} \qquad i = 1,2,\ldots,m \tag{56}$$

$$\mathrm{vec}(\mathbf{E}_{i}) \sim N\left\{0, \boldsymbol{\Sigma} = \sigma^{2}\left(\mathbf{Q}_{P} \otimes \mathbf{Q}_{K}\right)\right\} \tag{57}$$

where $\mathbf{E}_i$ is the random error matrix in normal distribution, $\boldsymbol{\Sigma}$ is the covariance matrix, $\mathbf{Q}_P$ is the cofactor matrix of the **p** points, $\mathbf{Q}_K$ is the cofactor matrix of the **k** coordinates of each point, $\otimes$ stands for the Kronecker product, and $\boldsymbol{\sigma}^2$ is the variance factor.

Let $\boldsymbol{\Sigma} = \sigma^{2}\mathbf{I}$

Least squares estimation of unknown transformation parameters $\mathbf{T}_i$, $\mathbf{c}_i$, and $\mathbf{t}_i$ (i=1,2,…,**m**) must satisfy the following objective function, as mentioned before in Equation (55),

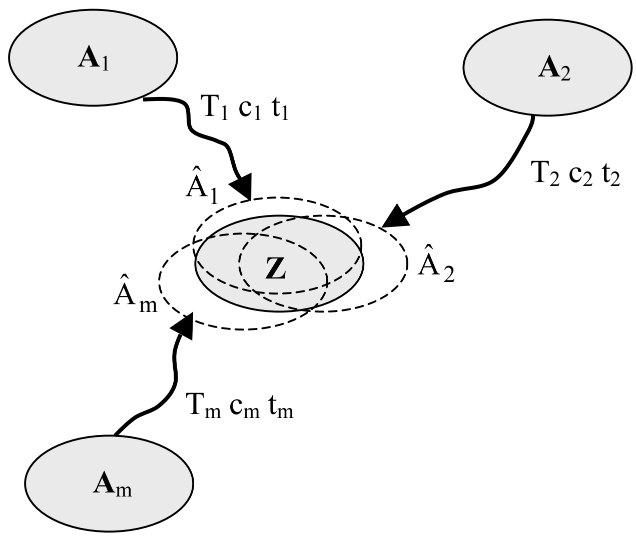

Figure 2: GP concept (Crosilla and Beinat, 2002)

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m} \left\| \hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j \right\|^2 = \min \tag{58}$$

Let us define a matrix **C** that is **geometrical centroid** of the transformed matrices, as follows:

$$\mathbf{C} = \frac{1}{m} \sum_{i=1}^{m} \hat{\mathbf{A}}_i \tag{59}$$

The following two objective functions

$$\sum_{i=1}^{m} \sum_{j=i+1}^{m} \left\| \hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j \right\|^2 = \sum_{i=1}^{m} \sum_{j=i+1}^{m} \operatorname{tr}\left\{ \left(\hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j \right)^T \left(\hat{\mathbf{A}}_i - \hat{\mathbf{A}}_j \right) \right\} \qquad 60)$$

$$m \sum_{i=1}^{m} \left\| \hat{\mathbf{A}}_i - \mathbf{C} \right\|^2 = m \sum_{i=1}^{m} \operatorname{tr}\left\{ \left(\hat{\mathbf{A}}_i - \mathbf{C} \right)^T \left(\hat{\mathbf{A}}_i - \mathbf{C} \right) \right\} \tag{61}$$

are equivalent (Kristof and Wingersky, 1971, Borg and Groenen, 1997). Therefore, Generalized Orthogonal Procrustes problem can also be solved minimizing Equation (61) instead of Equation (60). From a computational point of view, this solution method is simpler than the other one. Note that both of the solutions are iterative and equivalent, but two different ways.

In the following, only the solution method that imposes the minimum condition in Equation (61) will be expresses in detailed. The other solution that imposes the minimum condition in Equation (60) was proposed by Gower (1975), and improved by Ten Berge (1977).

The solution of the GP problem can be achieved using the following minimum condition

$$\sum_{i=1}^{m} \left\| \hat{\mathbf{A}}_i - \mathbf{C} \right\|^2 = \sum_{i=1}^{m} \mathrm{tr}\left\{ \left(\hat{\mathbf{A}}_i - \mathbf{C} \right)^T \left(\hat{\mathbf{A}}_i - \mathbf{C} \right) \right\} = \min \qquad (62)$$

in a iterative computation scheme of centroid $\mathbf{C}$ in a such a way:

Initialize:

- Define the initial centroid $\mathbf{C}$

Iterate:

- Direct solution of similarity transformation parameters of each model points matrix $\mathbf{A}_i$ with respect to the centroid $\mathbf{C}$ by means of Weighted Extended Orthogonal Procrustes (**WEOP**) solution
- After the calculation of each matrix $\hat{\mathbf{A}}_i$ is carried out, iterative updating of the centroid $\mathbf{C}$ according to Equation (59)

Until:

- Global convergence, i.e. stabilization of the centroid $\mathbf{C}$

The final solution for the centroid $\mathbf{C}$ shows the final coordinates of $\mathbf{p}$ points in the maximal agreement with respect to least squares objective function. Unknown similarity transformation parameters ($\mathbf{T}_i$, $\mathbf{c}_i$, and $\mathbf{t}_i$) can also be determined by means of WEOP calculation of each model points matrix $\mathbf{A}_i$ to the centroid $\mathbf{C}$.

The centroid $\mathbf{C}$ corresponds the least squares estimation $\hat{\mathbf{Z}}$ of the true value $\mathbf{Z}$. The proof of this definition was given by Crosilla and Beinat (2002).

$$\mathbf{C} = \hat{\mathbf{Z}} = \frac{1}{m} \sum_{i=1}^{m} \hat{\mathbf{A}}_i \qquad (63)$$

In the following parts of this section, different optional weighting strategies will be expressed. For further details and proof of the statements, author refers Crosilla and Beinat (2002), Beinat and Crosilla (2002).

Case 1:

Let us consider the following case,

$$\mathrm{vec}(\mathbf{E}_i) \sim N\left\{0, \boldsymbol{\Sigma} = \sigma^2 \left(\mathbf{Q}_P \otimes \mathbf{Q}_K \right) \right\} \quad , \quad \mathbf{Q}_K = \mathbf{I} \qquad (64)$$

where $\mathbf{Q}_P \neq \mathbf{I}$, but diagonal, i.e. each row of $\hat{\mathbf{A}}_i$ has different dispersion with respect to the true value $\mathbf{Z}$. But $\mathbf{Q}_P$ remains constant when varying $\mathbf{i}$=1,2,…,$\mathbf{m}$. In this case, centroid $\mathbf{C}$ is same as in Equation (59) or (63).

Case 2:

Let us treat a more general scheme,

$$\mathrm{vec}(\mathbf{E}_i) \sim N\left\{0, \mathbf{\Sigma}_i = \sigma^2 \left(\mathbf{Q}_{Pi} \otimes \mathbf{Q}_{Ki}\right)\right\} \quad , \quad \mathbf{Q}_{Ki} = \mathbf{I} \tag{65}$$

where $\mathbf{Q}_{Pi} \neq \mathbf{I}$, but diagonal, i.e. each row of $\hat{\mathbf{A}}_i$ has different dispersion with respect to the true value $\mathbf{Z}$ and the dispersion varies for each model points matrix **i**=1,2,…,**m**. In this case, the centroid $\mathbf{C}$ is defined as follow:

$$\mathbf{C} = \left(\sum_{i=1}^{m} \mathbf{P}_i\right)^{-1} \left(\sum_{i=1}^{m} \mathbf{P}_i \hat{\mathbf{A}}_i\right) \quad , \quad \mathbf{P}_i = \mathbf{Q}_{Pi}^{-1} \tag{66}$$

Also in this case, centroid $\mathbf{C}$ corresponds to the classical least squares estimation $\hat{\mathbf{Z}}$ of the true value $\mathbf{Z}$. Note that the imposed least squares objective function is

$$\sum_{i=1}^{m} \mathrm{tr}\left\{\left(\hat{\mathbf{A}}_i - \mathbf{Z}\right)^T \mathbf{P}_i \left(\hat{\mathbf{A}}_i - \mathbf{Z}\right)\right\} = \min \tag{67}$$

Case 3:

In real applications (for example block adjustment by independent models), all of the **p** points could not be visible in all of the model points matrices $\mathbf{A}_1$, $\mathbf{A}_2$, …, $\mathbf{A}_m$. In order to handle the *missing point* case, Commandeur (1991) proposed a method based on association to every matrix $\mathbf{A}_i$ a diagonal binary (**p** x **p**) matrix $\mathbf{M}_i$, in which the diagonal elements are 1 or 0 , according to existence or absence of the point in the **i**-th model (Figure 3). This solution can be considered as zero weights for the missing points.

$$\mathbf{A}_1 = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ - & - & - \\ - & - & - \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} - & - & - \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ - & - & - \\ x_6 & y_6 & z_6 \end{bmatrix} \qquad \mathbf{A}_3 = \begin{bmatrix} - & - & - \\ - & - & - \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{bmatrix}$$

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \mathbf{M}_2 = \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \quad \mathbf{M}_3 = \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

Figure 3: Incomplete $\mathbf{A}_i$ (**p** x **3**) model points matrices and resulting $\mathbf{M}_i$ (**p** x **p**) Boolean diagonal matrices (adapted from Beinat and Crosilla, 2001).

Least squares objective function and centroid $\mathbf{C}$ are as follows in the missing point case:

$$\sum_{i=1}^{m} \operatorname{tr}\left\{ \left[\left(c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T \right) - \mathbf{C} \right]^T \mathbf{M}_i \left[\left(c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T \right) - \mathbf{C} \right] \right\} = \min \tag{68}$$

or

$$\sum_{i=1}^{m} \operatorname{tr}\left\{ \left(\hat{\mathbf{A}}_i - \mathbf{C} \right)^T \mathbf{M}_i \left(\hat{\mathbf{A}}_i - \mathbf{C} \right) \right\} = \min \tag{69}$$

where

$$\mathbf{C} = \left(\sum_{i=1}^{m} \mathbf{M}_i \right)^{-1} \left[\sum_{i=1}^{m} \mathbf{M}_i \left(c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T \right) \right] \tag{70}$$

or

$$\mathbf{C} = \left(\sum_{i=1}^{m} \mathbf{M}_i \right)^{-1} \left[\sum_{i=1}^{m} \mathbf{M}_i \hat{\mathbf{A}}_i \right] \tag{71}$$

In order to obtain a more general scheme, one should consider the combined **weighted/missing** point solution. The weight matrix $\mathbf{P}_i$ and the binary matrix $\mathbf{M}_i$ can be combined in a product matrix, as follow:

$$\mathbf{D}_i = \mathbf{M}_i \mathbf{P}_i = \mathbf{P}_i \mathbf{M}_i \quad , \quad \mathbf{P}_i = \mathbf{Q}_{Pi}^{-1} \tag{72}$$

Note that $\mathbf{D}_i$ is also <u>diagonal</u>. In this case, the corresponding least squares objective function will be

$$\sum_{i=1}^{m} \operatorname{tr}\left\{ \left[\left(c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T \right) - \mathbf{C} \right]^T \mathbf{P}_i \mathbf{M}_i \left[\left(c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T \right) - \mathbf{C} \right] \right\} = \min \tag{73}$$

where the centroid $\mathbf{C}$ becomes

$$\mathbf{C} = \left(\sum_{i=1}^{m} \mathbf{P}_i \mathbf{M}_i \right)^{-1} \left[\sum_{i=1}^{m} \mathbf{P}_i \mathbf{M}_i \left(c_i \mathbf{A}_i \mathbf{T}_i + \mathbf{j}\mathbf{t}_i^T \right) \right] \tag{74}$$

Case 4:

The explained stochastic approaches up to this section deal different weighting strategies among the model points, not among the coordinate components. In order to account for the different accuracy of the tie-point coordinate components, Beinat and Crosilla (2002) proposed an anisotropic error condition.

$$\mathrm{vec}(\mathbf{E}_i) \sim N\{0, \mathbf{\Sigma}_i = \sigma^2(\mathbf{Q}_{Pi} \otimes \mathbf{Q}_{Ki})\} \quad , \quad \mathbf{Q}_{Pi} \neq \mathbf{I} \quad \text{and} \quad \mathbf{Q}_{Ki} \neq \mathbf{I} \tag{75}$$

where $\mathbf{Q}_{Pi}$ and $\mathbf{Q}_{Ki}$ are diagonal cofactor matrices. Then, weight matrices

$$\mathbf{P}_i = \mathbf{Q}_{Pi}^{-1} \tag{76}$$

and

$$\mathbf{K}_i = \mathbf{Q}_{Ki}^{-1} \tag{77}$$

The product matrix for the **weighted**/**missing** point solution is same as the previous definition,

$$\mathbf{D}_i = \mathbf{M}_i\mathbf{P}_i = \mathbf{P}_i\mathbf{M}_i \tag{78}$$

where $\mathbf{M}_i$ is the binary (Boolean) matrix. The corresponding least squares objective function will be

$$\sum_{i=1}^{m} \mathrm{tr}\left\{(\hat{\mathbf{A}}_i - \mathbf{C})^T \mathbf{D}_i (\hat{\mathbf{A}}_i - \mathbf{C})\mathbf{K}_i\right\} = \min \tag{79}$$

where the centroid $\mathbf{C}$ is

$$\mathrm{vec}(\mathbf{C}) = \left(\sum_{i=1}^{m} \mathbf{K}_i \otimes \mathbf{D}_i\right)^{-1} \left[\sum_{i}^{m} \mathbf{K}_i \otimes \mathbf{D}_i \mathrm{vec}(\hat{\mathbf{A}}_i)\right] \tag{80}$$

where centroid $\mathbf{C}$ corresponds to the classical least squares estimation $\hat{\mathbf{Z}}$ of the true value $\mathbf{Z}$. Note that $[\mathbf{K}_i \otimes \mathbf{D}_i]$ and $\mathrm{vec}(\hat{\mathbf{A}}_i)$ matrices are (**kp** x **kp**) and (**kp** x **1**) dimensional, respectively. For further details and the proof of the definition, author refers Beinat and Crosilla (2002).

2.6. Theoretical Precision for GP

Crosilla and Beinat (2002) gave the formulation of the a posteriori covariance matrix of the coordinates of each point as follow:

$$\mathbf{S}^j_{[k\times k]} = \frac{1}{m}\sum_{i=1}^{m} (\hat{\mathbf{A}}_i^T - \mathbf{C}^T)^j_{[k\times l]} (\hat{\mathbf{A}}_i - \mathbf{C})^j_{[l\times k]} \tag{81}$$

where $\mathbf{k}$ is the number of dimensions, $\mathbf{m}$ is the number of existence of the **j**-th point in the all models, and matrix $(\hat{\mathbf{A}}_i - \mathbf{C})^j_{[l\times k]}$ is the **k**-dimensional row vector for the **j**-th point in the **i**-th model points matrix. The off-diagonal elements of the $\mathbf{S}$ matrix show the algebraic correlation among the coordinate axes for the **j**-th point, not physical correlation.

3. APPLICATIONS IN PHOTOGRAMMETRY

As mentioned before in Section (2.5), the first step of Generalized Orthogonal Procrustes Analysis (GP) is definition of the initial centroid $\mathbf{C}$. One should define one of the models as fixed, and sequentially link the others by means of WEOP algorithm. Instead of sequentially registering pairs of single models, Beinat and Crosilla (2001) proposed the orientation of each model with respect to the topological union of all the previously oriented models. This process is shown schematically in Figure (4).

$$\mathbf{A}_2 \Rightarrow [\mathbf{A}_1]_{\text{fixed}}$$

$$\mathbf{A}_3 \Rightarrow \left[\tilde{\mathbf{A}}_2 \cup \mathbf{A}_1\right]$$

$$\mathbf{A}_4 \Rightarrow \left[\tilde{\mathbf{A}}_3 \cup \tilde{\mathbf{A}}_2 \cup \mathbf{A}_1\right] \qquad \ldots \text{ etc.}$$

Figure 4: Initial registration (adapted from Beinat and Crosilla, 2001)

The approximated shape of the whole object obtained in this way provides an initial value for the centroid $\mathbf{C}$. If the problem include the datum definition, e.g. in the case of block adjustment by independent models, a final WEOP is also needed to transform the whole object into the datum using ground control points.

In fact Generalized Orthogonal Procrustes Analysis (GP) is a **free** solution, since the consensus matrix $\mathbf{Z}$ is in any orientation-position-scale in the **k**-dimensional space. In other words, it does not involve the datum-constraints, e.g. ground control points. One of the most possible photogrammetric applications of the GP is block adjustment by independent models, which needs datum definition.

An adaptation of GP method into block adjustment by independent models problem was given by Crosilla and Beinat (2002).

At each iteration, all models $\hat{\mathbf{A}}_i$ of the block, one at a time, are rotated, translated and scaled to locally fit the temporary centroid $\mathbf{C}$ by using the WEOP and the common tie points existing between $\hat{\mathbf{A}}_i$ and $\mathbf{C}$. The centroid is computed from two sets of tie and control point coordinates together, all in the ground coordinate system. The control points, possibly with different weights, play the role of constraints in the centroid computation. They produce the same effect as *pseudo-observation equations* of the control point coordinates in the conventional solution of the block adjustment. During the adjustment, the centroid is not constant, but changes at each iteration because the tie point coordinates are constantly recomputed and updated, while the control point coordinates are kept fixed. As soon as a model $\hat{\mathbf{A}}_i$ is rotated, translated and scaled and its new coordinates stored, these changes are immediately applied and the centroid configuration is updated. The process ends when the centroid configuration variations between two subsequent iterations are smaller than a pre-defined threshold. This event means that the least squares fit among the models has been obtained (Crosilla and Beinat, 2002).

In the following parts of this section, 3 different examples will be given in order to compare the Procrustes method with the conventional least-squares adjustment. All of the examples were performed on a PC that has the following specifications: Windows 2000 Professional OS, Intel Pentium III 450 Mhz CPU, and 128 MB RAM.

3.1. Example 1

At the first attempt, conventional least-squares adjustment for similarity transformation and WEOP were compared according to their computational expense. The problem is the least-squares estimation of the similarity transformation parameters between two model point matrices, as mentioned before in Section (2.4).

A synthetic model points matrix **A**, in which are 100 points in 3-dimensional space, and its transformed counterpart **B** was generated. Additionally, the coordinate values of the matrix **A** were disturbed by the random error **e** that is in the following distribution,

$$e \sim N\{\mu = 0,\ \sigma = \pm 5\text{mm}\} \tag{82}$$

The computation times were given in Table (1). As mentioned before, Weighted Extended Orthogonal Procrustes (WEOP) solution is a direct solution as opposed to the conventional least-squares solution. The initial approximations of the unknowns were calculated using a closed-form solution proposed by Dewitt (1996), since the functional model of the conventional least-squares solution for this problem is not linear.

	Iterations	Computation time (sec.)
Least-squares adjustment	3	0.09
WEOP	--	0.03

Table 1: Conventional least-squares solution versus WEOP.

Of course, same results for the unknown transformation parameters **T**, **c**, **t** were obtained in both solutions, in spite of two different solution ways. In WEOP solution, the core of the computation is Singular Value Decomposition of the (**k** x **k**) matrix, in this example it is (3 x 3). The used solution strategy in conventional least-squares solution is well-known method, i.e. normal matrix partitioning by means of groups of the unknown, Cholesky decomposition, and back-substitution. Note that the coordinates of the control points were treated as stochastic quantities with proper weights.

$$\begin{aligned} v_L &= A_1 \cdot t + A_2 \cdot x - \ell_L \quad ; \quad P_L \\ v_C &= \qquad\qquad I \cdot x - \ell_C \quad ; \quad P_C \end{aligned} \tag{83}$$

where **t** and **x** are unknown vectors of absolute orientation parameters and object space coordinates, respectively. Therefore, the dimension of the normal equations matrix in this example was [(**7** + **pk**) x (**7** + **pk**)], namely (307 x 307).

3.2. Example 2

In the second example, a real data set, which consisted 5 model points matrices obtained from a close-range laser scanner device, was used. The data set includes totally 10 tie points in the 3-dimensional space, also in unit of meter. The expected a priori precision of the coordinate observations is $\sigma_0 = \pm 3_{mm}$ along the 3-coordinate axes. The aim is to combine all models into a common coordinate system in order to obtain the whole object boundary. Two different methods were employed in order to achieve the solution; block adjustment by independent models as conventional least-squares solution, and/versus Generalized Orthogonal Procrustes method (GP). Table (2) shows the result.

	Iterations	Computation times (sec.)	σ0 (mm.)
Block adjustment by independent models	3	0.01	3.4
Generalized Orthogonal Procrustes (GP)	6	0.01	2.2

Table 2: Block adjustment by independent models versus Generalized Orthogonal Procrustes (GP).

In Table (2) **σ0** value of GP method was calculated according to the deviations of the transformed coordinates from the final centroid **C**. In block adjustment by independent models method, same solution strategy mentioned in Section (3.1) was employed. Three of the tie points were involved as control points in order to define the datum using the same functional model in Equation (83). In contrary, Generalized Orthogonal Procrustes (GP) solution is completely free solution. In other word, it does not involve any object space constraint. This circumstance is also the reason of slight difference between the two **σ0** values.

One of the most important advantage of the GP method against to block adjustment by independent models method is its drastically less memory requirement. Required basic memory sizes for this example are given in the following part. Note that the variables are double precision, e.g. 8 bytes.

For block adjustment by independent models method:

For $\mathbf{N}_{11}$:	**m** (**u** x **u**)	= 5 . (7 x 7)	= 245 variables
For $\mathbf{N}_{12}$:	(**m u**) x (**p k**)	= (5 .7) x (10 . 3)	= 1050 variables
For $\mathbf{N}_{22}$:	**p k**	= 10 . 3	= 30 variables
Totally :			= **10 600 Bytes**

For Generalized Orthogonal Procrustes (GP) method:

For unknowns of each model :	**m u**	= 5 . 7	= 35 variables
For centroid **C** :	**p k**	= 10 . 3	= 30 variables
Totally :			= **520 Bytes**

where $\mathbf{N}_{11}$, $\mathbf{N}_{12}$, and $\mathbf{N}_{22}$ are the partitioned sub-parts of the normal equations matrix, **m** is number of the models, **p** is number of points, **k** is number of dimensions , and **u** is number of unknown transformation parameters for a model.

3.3. Example 3

In the last example, a synthetic data set, which consisted 9 model points matrices, is used. The data set includes totally 100 tie points, in which 30 of them are control points, in the 3-dimensional space. The data set was slightly disturbed by the following random error **e**:

$$e \sim N\{\mu = 0,\ \sigma = \pm 0.002\ _{\text{unitless}}\} \tag{84}$$

Table (3) shows the calculation information of the two methods, i.e. block adjustment by independent models as conventional least-squares solution, and/versus Generalized Orthogonal Procrustes method (GP). In both methods, control points were employed as datum-definitions. In GP method, the control points were treated as in the method, which adapts the GP method to block adjustment by independent models (Crosilla and Beinat, 2002), as expressed in Section (3).

	Iterations	Computation times (sec.)	σX (unitless)	σY (unitless)	σZ (unitless)
Block adjustment by independent models	5	1.032	0.0018	0.0018	0.0019
Generalized Orthogonal Procrustes (GP)	35	1.953	0.0017	0.0020	0.0019

Table 3: Block adjustment by independent models versus Generalized Orthogonal Procrustes (GP).

As mentioned before in Section (3.1), the most computationally expensive part of the Procrustes method is Singular Value Decomposition of the (**k** x **k**) matrix, in this example it is (3 x 3). The used solution strategy in conventional least-squares solution is well-known method, i.e. normal matrix partitioning by means of groups of the unknown, Cholesky decomposition, and back-substitution. Note that the coordinates of the control points were treated as stochastic quantities with proper weights.

In the case of datum-definition, very slow convergence behavior of the Generalized Orthogonal Procrustes (GP) method compared to conventional block adjustment solution can be shown from Table (3).

3.4. Comparison of the two methods

- The Procrustes analysis is a linear least-squares solution to compute the similarity transformation parameters among the **m** $(m \geq 2)$ model points matrices in **k**-dimensional space. Since its functional model is linear, it does not need initial approximations for the unknown similarity transformation parameters. But in the case of conventional least-squares adjustment, the linearisation and initial approximations of the unknowns in the case of 3 or more dimensional similarity transformations are needed due to non-linearity of the functional model. In the literature, there are many closed-form solutions to calculate the initial approximations for the unknown similarity transformation parameters (Thompson, 1959, Schut 1960, Oswal, Balasubramanian, 1968, Dewitt, 1996).

- The Procrustes analysis does not has a restriction on the number of **k** dimensions in the space of the data set. Its generic and flexible functional model can easily handle the **k** $(k > 3)$ dimensional similarity transformation problems without any arrangement on the mathematical model. In photogrammetry area, we are very familiar to **k** $(k = 2,3)$ dimensional similarity transformations. In the case of $\mathbf{k} > 3$ dimensional similarity transformation problems, the functional model of the conventional least squares adjustment must be extended/rearranged according to the number of dimensions of the data set.

- The Generalized Orthogonal Procrustes analysis (GP) is a free solution, in other words, it does not involve the control information to define the datum, except the adaptation to block adjustment by independent models (Crosilla and Beinat, 2002). This configuration can also be achieved in the conventional least-squares adjustment by means of inner constraints, or sometimes referred as free net adjustment.

- For the time being, no work has been reported on the most general stochastic model, namely existence of correlation among the all measurements, for Procrustes analysis. From the mathematical point of view, conventional least squares adjustment has very powerful mathematical (functional + stochastic) model, which can handle many physically real situations, e.g. unknowns as stochastic quantities, constraints among the measurements and among the unknowns, correlated measurements, etc…

- In the Procrustes analysis, the most computationally expensive part of the calculation is Singular Value Decomposition of the (**k** x **k**) matrix, where **k** is the number of dimensions of the data set. But it's relatively slow convergence behavior makes its computation speed equal with compared to conventional least-squares adjustment.

- From the software implementation point of view, the Procrustes analysis needs drastically less memory requirement than the conventional least-squares adjustment, as explained by a simple example in Section (3.2.).

- The Procrustes method does not has any reliability criterion in order to detect and localize the blunders, although this feature is vital for the real applications, in which measurements might include the blunders. The conventional least-squares adjustment has many powerful tools in order to localize and eliminate the blunders, e.g. Data-Snooping and Robust methods.

4. CONCLUSIONS

The Procrustes analysis is a least-squares method to estimate the unknown similarity transformation parameters among two or more than two model points matrices up to their maximal agreement. Because the estimation model is linear, it does not require the initial approximations of the unknowns. In geodetic sciences, we are very familiar to solve the $\mathbf{k} = 2, 3$ dimensional similarity transformations by means of conventional least-squares adjustment. In fact, these two different methods offer two different ways to achieve the same solution.

In this report, a survey on Procrustes analysis, its theory, algorithms, and related works has been given. Also, its applications in photogrammetry has been addressed. The previous section (3.4.) gives a comparison between the Procrustes analysis and the conventional least squares adjustment.

The most important disadvantage of the Procrustes method is lack of reliability criterion in order to detect and localize the blunders, which might be included by the data set. Without such a tool, the results that produced by the Procrustes method can be wrong in the case of existence of blunders in the data set.

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NOTE

In order to perform the experimental part of this semester Praktikum, two programs, i.e. block adjustment by independent models and the generalized Procrustes analysis (GP), were developed as ANSII C++ classes by the author, and are available in the internal Web area of Chair of Photogrammetry and Remote Sensing.

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